
MANIN'S CONJECTURE FOR QUARTIC DEL PEZZO SURFACES WITH A CONIC FIBRATION

by

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Abstract. — An asymptotic formula is established for the number of \mathbb{Q} -rational points of bounded height on a non-singular quartic del Pezzo surface with a conic bundle structure.

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1. Introduction

Let k be a number field. This investigation centres upon the distribution of k -rational points on conic bundle surfaces X/\mathbb{P}_k^1 . These are defined to be projective non-singular surfaces X defined over k , which are equipped with a dominant k -morphism $\pi : X \rightarrow \mathbb{P}_k^1$, all of whose fibres are conics. A summary of our knowledge concerning the arithmetic of conic bundle surfaces can be found in the article by Colliot-Thélène [9]. If $-K_X$ denotes the anticanonical divisor of X , then the degree of X is defined

to be the self-intersection number $(-K_X, -K_X) = 8 - r$, where r is the number of geometric fibres above π that are degenerate. When $0 \leq r \leq 3$ it is known that the Hasse principle holds for these surfaces, and furthermore, that such X are k -rational as soon as they possess k -rational points. When $r = 4$, so that X has degree 4, it has been shown by Iskovskikh [17, Proposition 1] that two basic cases arise. Either $-K_X$ is not ample, in which case X is k -birational to a generalised Châtelet surface, or else $-K_X$ is ample, in which case X is a non-singular quartic del Pezzo surface.

Our interest lies with the quantitative arithmetic of degree 4 conic bundle surfaces. When $X(k) \neq \emptyset$ and the anticanonical height function $H : X(k) \rightarrow \mathbb{R}_{\geq 0}$ is associated, we seek to determine the asymptotic behaviour of the counting function

$$N_{U,H}(B) := \#\{x \in U(k) : H(x) \leq B\},$$

as $B \rightarrow \infty$, for a suitable Zariski open subset $U \subseteq X$. The conjecture that drives our work is due to Manin [13]. Let $\text{Pic } X$ be the Picard group of X . Still under the assumption that $X(k)$ is non-empty, this conjecture predicts the existence of a positive constant $c_{X,H}$ such that

$$N_{U,H}(B) = c_{X,H} B (\log B)^{\text{rank}(\text{Pic } X) - 1} (1 + o(1)), \quad (1.1)$$

as $B \rightarrow \infty$. Peyre [22] has given a conjectural interpretation of $c_{X,H}$ in terms of the geometry of X . Until very recently we were not in possession of a single conic bundle surface of degree 4 for which this refined conjecture could be established.

Henceforth we will be interested in the case $k = \mathbb{Q}$. In joint work with Peyre [6], the authors have made a study of generalised Châtelet surfaces, ultimately establishing (1.1) for a family of such surfaces that fail to satisfy weak approximation. The aim of the present investigation is to produce a satisfactory treatment of a non-singular del Pezzo surface of degree 4 with a conic bundle structure which admits a section over \mathbb{Q} . Such surfaces are defined as the intersection of two quadrics in $\mathbb{P}_{\mathbb{Q}}^4$. When $X(\mathbb{Q}) \neq \emptyset$, we may assume that $X \subset \mathbb{P}_{\mathbb{Q}}^4$ is cut out by the system

$$\begin{cases} \Phi_1(x_0, \dots, x_4) := x_0x_1 - x_2x_3 = 0, \\ \Phi_2(x_0, \dots, x_4) = 0, \end{cases} \quad (1.2)$$

for quadratic forms $\Phi_1, \Phi_2 \in \mathbb{Z}[x_0, \dots, x_4]$ such that the Jacobian matrix $(\nabla \Phi_1, \nabla \Phi_2)$ has full rank throughout X .

Let $\|\cdot\| : \mathbb{R}^5 \rightarrow \mathbb{R}_{\geq 0}$ be a norm. Given a point $x = [\mathbf{x}] \in \mathbb{P}_{\mathbb{Q}}^4(\mathbb{Q})$, with $\mathbf{x} = (x_0, \dots, x_4) \in \mathbb{Z}^5$ such that $\gcd(x_0, \dots, x_4) = 1$, we let $H(x) := \|\mathbf{x}\|$. Then H is the anticanonical height metrized by the choice of norm. Any line contained in X that is defined over \mathbb{Q} will automatically contribute $cB^2 + O(B \log B)$ to $N_{U,H}(B)$, for an appropriate constant $c > 0$. Hence it is natural to take $U \subset X$ to be the open subset formed by deleting the 16 lines from X .

The best evidence that we have for (1.1) in the setting of non-singular surfaces of the shape (1.2) is due to Salberger. In work communicated at the conference ‘‘Higher dimensional varieties and rational points’’ at Budapest in 2001, he establishes the upper bound

$$N_{U,H}(B) = O_X(B^{1+\varepsilon}), \quad (1.3)$$

for any $\varepsilon > 0$. Here, as throughout our work, we allow the implied constant to depend on the choice of ε . The Manin conjecture has received a great deal of attention in the context of singular del Pezzo surfaces of degree 3 and 4. An account of recent progress can be found in the second author's survey [7]. There is general agreement among researchers that the level of difficulty in establishing the expected asymptotic formula for del Pezzo surfaces increases as the degree decreases or as the singularities become milder. Among the non-singular del Pezzo surfaces, those of degree at least 6 are all toric and so are handled by the work of Batyrev and Tschinkel [1]. In [3] the first author gave the earliest satisfactory treatment of a non-singular del Pezzo surface of degree 5. As highlighted by Swinnerton-Dyer [25, Question 15], it has become something of a milestone to establish the Manin conjecture for a single non-singular del Pezzo surface of degree 3 or 4.

In this paper we will be concerned with a quartic del Pezzo surface $X \subset \mathbb{P}_{\mathbb{Q}}^4$ of the shape (1.2), with

$$\Phi_2(\mathbf{x}) := x_0^2 + x_1^2 + x_2^2 - x_3^2 - 2x_4^2. \quad (1.4)$$

In particular X contains obvious lines defined over \mathbb{Q} , from which it follows that X is \mathbb{Q} -rational. This fact is recorded by Colliot-Thélène, Sansuc and Swinnerton-Dyer [10, Proposition 2], for example. If L is a line in X defined over \mathbb{Q} then it is a simple consequence of the fact that any plane through L must cut out a pair of lines L, L_i on each quadric $\Phi_i = 0$ defining X , and the intersection $L_1 \cap L_2$ meets X in exactly one further point, which is defined over \mathbb{Q} .

Let $\|\cdot\|$ be the norm on \mathbb{R}^5 given by

$$\|\mathbf{x}\| := \max \left\{ |x_0|, |x_1|, |x_2|, |x_3|, \sqrt{\frac{2}{3}}|x_4| \right\}. \quad (1.5)$$

All that is required of this norm is that $\|\mathbf{x}\| = \max\{|x_0|, |x_1|, |x_2|, |x_3|\}$ for every $\mathbf{x} = (x_0, \dots, x_4) \in \mathbb{R}^5$ such that $[\mathbf{x}] \in X$, and furthermore $\|\mathbf{x}^\sigma\| = \|\mathbf{x}\|$, where \mathbf{x}^σ is the vector obtained by permuting the variables x_0, \dots, x_3 and leaving x_4 fixed. It would be possible to work instead with the norm $|\mathbf{x}| := \max\{|x_0|, \dots, |x_4|\}$, but not without introducing extra technical difficulties that we wish to suppress in the present investigation. We are now ready to reveal our main result.

Theorem. — *We have*

$$N_{U,H}(B) = c_{X,H} B(\log B)^4 + O\left(\frac{B(\log B)^4}{\log \log B}\right),$$

where $c_{X,H} > 0$ is the constant predicted by Peyre.

During the final preparation of this paper, the authors have learnt of independent work by Fok-Shuen Leung [18] on the conic bundle surface given by (1.2) and (1.4). This sharpens Salberger's estimate in (1.3), ultimately providing upper and lower bounds for $N_{U,H}(B)$ that are of the expected order of magnitude. Our result supersedes this, and confirms the estimate predicted by Manin and Peyre in (1.1) for the non-singular quartic del Pezzo surface under consideration. The fact that the Picard group has rank 5, as needed to verify the power of $\log B$, will be established in due course.

The proof of our theorem is long and complicated. We therefore dedicate the remainder of this introduction to surveying some of its key ingredients and indicating some obvious lines for further enquiry. Given any non-singular surface defined by the system (1.2), it is possible to define a pair of conic bundle morphisms $f_i : X \rightarrow \mathbb{P}_{\mathbb{Q}}^1$, for $i = 1, 2$. Specifically, for any $x \in X$, one takes

$$f_1(x) = \begin{cases} [x_0, x_2], & \text{if } (x_0, x_2) \neq (0, 0), \\ [x_3, x_1], & \text{if } (x_1, x_3) \neq (0, 0), \end{cases}$$

and

$$f_2(x) = \begin{cases} [x_0, x_3], & \text{if } (x_0, x_3) \neq (0, 0), \\ [x_2, x_1], & \text{if } (x_1, x_2) \neq (0, 0). \end{cases}$$

For a given point $x \in X(\mathbb{Q})$ of height $H(x) \leq B$, it follows from the general theory of height functions that there exists an index i such that $x \in f_i^{-1}(t)$ for some $t \in \mathbb{P}_{\mathbb{Q}}^1(\mathbb{Q})$ of height $O(B^{1/2})$. The idea is now to count rational points of bounded height on the fibres $f_1^{-1}(t)$ and $f_2^{-1}(t)$, uniformly for points $t \in \mathbb{P}_{\mathbb{Q}}^1(\mathbb{Q})$ of height $O(B^{1/2})$. This is the strategy adopted by Salberger in his proof of (1.3).

In the present situation, with the quadratic form (1.4), the fibres that we need to examine have the shape

$$C_{a,b} : (a^2 - b^2)x^2 + (a^2 + b^2)y^2 = 2z^2, \quad (1.6)$$

for coprime $a, b \in \mathbb{Z}$. It is clear that $C_{a,b} \subset \mathbb{P}_{\mathbb{Q}}^2$ is a non-singular plane conic when the discriminant $\Delta(a, b) = -2(a^4 - b^4)$ is non-zero. The reduction of the counting problem to one involving the family of conics (1.6) is carried out in §3, where we have avoided using the height machinery by doing things in a completely explicit manner.

As is well-known there is a group homomorphism $\text{Pic } X \rightarrow \mathbb{Z}$, which to a divisor class $D \in \text{Pic } X$ associates the intersection number of D with a fibre. The kernel of this map is generated by the “vertical” divisors, which up to linear equivalence are the irreducible components of the fibres. Since the non-singular fibres are all linearly equivalent, it follows that $\text{Pic } X$ has rank $2 + n$, where n is the number of split singular fibres above closed points of $\mathbb{P}_{\mathbb{Q}}^1$. In our case there are three closed points, corresponding to the irreducible factors $a - b, a + b$ and $a^2 + b^2$ of $\Delta(a, b)$. Since each singular fibre is split it follows that $\text{Pic } X \cong \mathbb{Z}^5$, as previously claimed.

Roughly speaking, as one ranges over values of $[a, b] \in \mathbb{P}_{\mathbb{Q}}^1(\mathbb{Q})$ of height $O(B^{1/2})$, one expects there to be about $(\log B)^4$ rational points on $C_{a,b}$ with height restricted to appropriate intervals. The preliminary reduction to $[a, b] \in \mathbb{P}_{\mathbb{Q}}^1(\mathbb{Q})$ of height $O(B^{1/2})$ is absolutely pivotal here: it is only through this device that we can cover $U(\mathbb{Q})$ with a satisfactory number of divisors. Were we charged instead with establishing an upper bound like (1.3), our analysis would now be relatively straightforward, thanks to the control over the growth rate of rational points on conics afforded by the second author’s joint work with Heath-Brown [8, Theorem 6]. A key aspect of this estimate is that it is uniform in the height of the conic, becoming sharper as the discriminant grows larger. In §4 we will take advantage of these arguments to eliminate certain awkward ranges for a, b in (1.6).

Obtaining an asymptotic formula is a far more exacting task. Using the large sieve inequality, Serre [24] has shown that most plane conics defined over \mathbb{Q} don't contain rational points. This phenomenon might pose problems for us, given that we want a uniform asymptotic formula for a Zariski dense set of rational points on the fibres. Our choice of surface has been tailored to guarantee that this doesn't happen, since the corresponding fibres (1.6) always contain the rational point

$$\xi = [1, -1, a]. \quad (1.7)$$

In the classical manner we can use this point to parametrise all of the rational points on the conic, which ultimately leads us to evaluate asymptotically the number of points belonging to a 2-dimensional sublattice $\Lambda \subset \mathbb{Z}^2$ which are constrained to lie in an appropriate region $\mathcal{R} \subset \mathbb{R}^2$. Both Λ and \mathcal{R} depend on the parameters a and b , so this estimate needs to be achieved with a sufficient degree of uniformity. Assuming that \mathcal{R} has piecewise continuous boundary, we would ideally like to apply the familiar estimate

$$\#(\Lambda \cap \mathcal{R}) = \frac{\text{vol}(\mathcal{R})}{\det \Lambda} + O(\partial \mathcal{R} + 1), \quad (1.8)$$

where $\partial \mathcal{R}$ denotes the perimeter of \mathcal{R} . Unhappily this estimate is too crude for our purposes. We will use Poisson summation to make the error term explicit. The reduction of the problem to a lattice point counting problem is carried out in §5 and its execution is the subject of §§6–8.

The one outstanding task, which is the focus of §9, is to evaluate asymptotically the main term arising from the lattice point counting problem. It turns out that this involves a sum of the shape

$$\sum_{\substack{(a,b) \in \mathbb{Z}^2 \cap \mathcal{R} \\ \gcd(a,b)=1}} \frac{g(|a^4 - b^4|)}{\max\{a, b\}^2},$$

for a certain multiplicative arithmetic function g that is very similar to the ordinary divisor function $\tau(n) := \sum_{d|n} 1$. The problem of determining the average order of the divisor function as it ranges over the values of polynomials has enjoyed considerable attention in the literature. In the setting of binary forms current technology has limited us to handling forms of degree at most 4. When $g = \tau$ Daniel [11] has dealt with the case of irreducible binary quartic forms. We need to extend this argument to deal with a more general class of arithmetic functions and to binary quartic forms that are no longer irreducible. We have found it convenient to corral the necessary estimates into a separate investigation [5], which is of a more technical nature. The facts that we will need are recalled in §2.

Our main goal in this paper is to outline a general strategy for proving the Manin conjecture for non-singular del Pezzo surfaces of degree 4 equipped with a conic bundle structure admitting a section over \mathbb{Q} . In order to minimise the length and technical difficulty of the work, which is already considerable, we have chosen to illustrate the approach by selecting a concrete surface to work with. It is likely that the argument we present can be adapted to handle other conic bundle surfaces. For example, consider

the family of surfaces (1.2), with

$$\Phi_2(\mathbf{x}) := c_0x_0^2 + c_1x_1^2 + c_2x_2^2 + c_3x_3^2 + c_4x_4^2.$$

It is easy to check that X will be non-singular if and only if $c_0 \cdots c_4 \neq 0$ and $c_0c_1 \neq c_2c_3$. As we've already mentioned, our success with (1.4) is closely linked to the existence of an obvious rational point (1.7) on all of the fibres (1.6). This is equivalent to the map $X(\mathbb{Q}) \rightarrow \mathbb{P}_{\mathbb{Q}}^1(\mathbb{Q})$ being surjective. A necessary and sufficient condition for ensuring this is that the morphism $X \rightarrow \mathbb{P}_{\mathbb{Q}}^1$ should admit a section over \mathbb{Q} . This fact has a long history and can be traced back to work of Lewis and Schinzel [19]. Even when the conic bundle surface admits a section, however, there remains considerable work to be done in handling a general diagonal form Φ_2 as above. The corresponding fibres will take the shape

$$f_1(a, b)x^2 + f_2(a, b)y^2 + c_4z^2 = 0,$$

for binary forms $f_1(a, b) = c_0a^2 + c_3b^2$ and $f_2(a, b) = c_2a^2 + c_1b^2$. Thus, at the very least, one needs analogues of our investigation [5] for the case in which f_1f_2 factors over \mathbb{Q} as the product of two quadratic forms or splits completely.

Notation. — Throughout our work \mathbb{N} will denote the set of positive integers. If $a, b \in \mathbb{N}$ then we write $\gcd(a, b)$ for the greatest common divisor of a, b and $[a, b] = ab/\gcd(a, b)$ for the least common multiple. We set

$$(a, b)_b := 2^{-\nu_2(\gcd(a, b))} \gcd(a, b) \quad (1.9)$$

for the odd part of the greatest common divisor, and use the symbol \sum^b to indicate a summation in which all the variables of summation are restricted to odd integers. Furthermore, our work will involve the arithmetic functions

$$\varphi(n) := n \prod_{p|n} \left(1 - \frac{1}{p}\right), \quad \varphi^*(n) := \frac{\varphi(n)}{n}, \quad \varphi^\dagger(n) := \prod_{p|n} \left(1 + \frac{1}{p}\right). \quad (1.10)$$

Finally, in terms of the parameter B , we set

$$Z_1 := B^{1/\log \log B}, \quad Z_2 := \log \log B. \quad (1.11)$$

We will reserve $c > 0$ for a generic absolute positive constant, whose value is always effectively computable.

Acknowledgements. — This investigation was undertaken while the second author was visiting the *Université Paris 6 Pierre et Marie Curie* and the *Université Paris 7 Denis Diderot*. The hospitality and financial support of these institutions is gratefully acknowledged. It is a pleasure to thank Olivier Wittenberg for useful conversations relating to the geometry of conic bundle surfaces and the anonymous referee for numerous pertinent remarks. While working on this paper the second author was supported by EPSRC grant number EP/E053262/1.

2. Technical results

Our work requires a number of auxiliary results, ranging from basic estimates using the geometry of numbers to more sophisticated results concerning the divisor problem for binary quartic forms.

2.1. Counting rational points on curves. — As made clear in the introduction to this paper, our proof of the theorem uses the conic bundle structure in order to focus the effort on a family of curves of low degree and rather low height. The following result is due to Heath-Brown [16, Lemma 3], and deals with the situation for lines in $\mathbb{P}_{\mathbb{Q}}^2$, the rational points on which basically correspond to integer lattices of rank 2.

Lemma 1. — *Let $\Lambda \subseteq \mathbb{Z}^2$ be a lattice of rank 2 and determinant $\det \Lambda$, and let $\mathcal{R} \subset \mathbb{R}^2$ be any region with piecewise continuous boundary. Then we have*

$$\#\{\mathbf{x} \in \Lambda \cap \mathcal{R} : \gcd(x_1, x_2) = 1\} \ll 1 + \frac{\text{vol}(\mathcal{R})}{\det \Lambda}.$$

Our next uniform upper bound is extracted from joint work of the second author with Heath-Brown [8, Corollary 2], and handles the case of non-singular plane conics.

Lemma 2. — *Let $C \subset \mathbb{P}_{\mathbb{Q}}^2$ be a non-singular conic. Assume that the underlying quadratic form has matrix of determinant Δ , and that the 2×2 minors have greatest common divisor Δ_0 . Then we have*

$$\#\left\{\mathbf{x} \in C \cap \mathbb{Z}^3 : \begin{array}{l} \gcd(x_1, x_2, x_3) = 1, \\ |x_i| \leq B_i, (1 \leq i \leq 3) \end{array} \right\} \ll \tau(|\Delta|) \left(1 + \frac{B_1 B_2 B_3 \Delta_0^{3/2}}{|\Delta|}\right)^{1/3}.$$

Taking $B_1 = B_2 = B_3 = B$ in Lemma 2, we retrieve the well-known fact that a non-singular plane conic $C \subset \mathbb{P}_{\mathbb{Q}}^2$ contains $O_C(B)$ rational points of height B .

2.2. Generalisation of Nair's lemma. — In the setting of polynomials in only one variable there is a well-known result due to Nair [21] which provides upper bounds for the average order of suitable non-negative arithmetic functions as they ranges over the values of the polynomial. Taking advantage of the authors' refinement [4] of this work we have the following result.

Lemma 3. — *Let $\varepsilon > 0$, let $a \in \mathbb{N}$ and let $\delta \in \{0, 1\}$. Then for $x \gg a^\varepsilon$ there exists an absolute constant $c > 0$ such that*

$$\sum_{n \leq x} \tau(n)^\delta \tau(n^2 + a) \ll \varphi^\dagger(a)^c x (\log x)^{1+\delta}.$$

Proof. — Define the multiplicative arithmetic function

$$\tau'(p^\nu) := \begin{cases} 2, & \text{if } \nu = 1, \\ (1 + \nu)^2, & \text{if } \nu \geq 2, \end{cases}$$

for any prime power p^ν . We have $\tau(n_1)\tau(n_2) \leq \tau'(n_1n_2)$ for any $n_1, n_2 \in \mathbb{N}$. Let $S_\delta(x)$ denote the sum that is to be estimated. On replacing τ by τ' we find that

$$S_\delta(x) \leq \sum_{n \leq x} \tau'(n^\delta(n^2 + a)).$$

Now for any $a \in \mathbb{N}$ it is clear that the polynomial $f(t) = t^\delta(t^2 + a)$ has degree $2 + \delta$, that it has discriminant $\Delta_f = -4a^{1+2\delta}$ and that it has no fixed prime divisor if $\delta = 0$ or a is even. If $\delta = 1$ and a is odd then 2 is a fixed prime divisor of f . But then we may break the sum into two sums according to whether n is even or odd and make a corresponding change of variables, absorbing the additional factor $\tau'(2) = 2$ into an implied constant. An application of [4, Theorem 2] now reveals that

$$S_\delta(x) \ll x \prod_{p \leq x} \left(1 - \frac{\varrho_f(p)}{p}\right) \sum_{m \leq x} \frac{\tau'(m)\varrho_f(m)}{m},$$

for $x \gg a^\varepsilon$, where $\varrho_f(m)$ denotes the number of roots modulo m of the congruence $f(n) \equiv 0 \pmod{m}$. Recall the well-known inequalities $\varrho_f(p^\nu) \leq 3p^{2\nu_p(\Delta_f)}$ for any prime power, and $\varrho_f(p^\nu) \leq 3$ if $p \nmid \Delta_f$. Then it follows that

$$\begin{aligned} \sum_{m \leq x} \frac{\tau'(m)\varrho_f(m)}{m} &\leq \prod_{p \leq x} \left(1 + \frac{\tau'(p)\varrho_f(p)}{p} + \sum_{\nu \geq 2} \frac{\tau'(p^\nu)\varrho_f(p^\nu)}{p^\nu}\right) \\ &\leq \exp\left(\sum_{p \leq x} \frac{\tau'(p)\varrho_f(p)}{p} + \sum_{p \leq x} \sum_{\nu \geq 2} \frac{\tau'(p^\nu)\varrho_f(p^\nu)}{p^\nu}\right) \\ &\ll \exp\left(\sum_{p \leq x} \frac{2\varrho_f(p)}{p} + \sum_{p^\sigma \parallel 4a^{1+2\delta}} \sum_{\nu \geq 2} \frac{3(1+\nu)^2 p^{2(1+2\delta)\sigma}}{p^\nu}\right) \\ &\ll \varphi^\dagger(a)^c \exp\left(\sum_{p \leq x} \frac{2\varrho_f(p)}{p}\right), \end{aligned}$$

for a suitable absolute constant $c > 0$. On noting that

$$\varrho_f(p) = 1 + \delta + \left(\frac{-a}{p}\right),$$

for $p > 2$, this therefore concludes the proof of the lemma. \square

We will also need a version of Nair's lemma for binary forms $F \in \mathbb{Z}[x_1, x_2]$. Let $\|F\|$ denote the maximum modulus of its coefficients and set

$$\varrho_F^*(m) := \frac{1}{\varphi(m)} \#\left\{(n_1, n_2) \in (0, m]^2 : \begin{array}{l} \gcd(n_1, n_2, m) = 1 \\ F(n_1, n_2) \equiv 0 \pmod{m} \end{array}\right\},$$

for any $m \in \mathbb{N}$. The following estimate is deduced from [4, Corollary 1] by specialising to the generalised divisor function $\tau_k(n) := \sum_{n=d_1 \cdots d_k} 1$.

Lemma 4. — *Let $\varepsilon > 0$ and let $F \in \mathbb{Z}[x_1, x_2]$ be a non-zero binary form of degree d , with $\text{Disc}(F) \neq 0$ and $F(1, 0)F(0, 1) \neq 0$. Then we have*

$$\sum_{|n_1| \leq B_1} \sum_{|n_2| \leq B_2} \tau_k(|F(n_1, n_2)|) \ll_{d,k} \|F\|^\varepsilon \left(B_1 B_2 E + \max\{B_1, B_2\}^{1+\varepsilon}\right),$$

where

$$E := \prod_{d < p \leq \min\{B_1, B_2\}} \left(1 + \frac{\varrho_F^*(p)(k-1)}{p}\right).$$

2.3. The divisor problem for binary forms. — Throughout this section we let i denote a generic element from the set $\{1, 2\}$. Let $L_i, Q \in \mathbb{Z}[x_1, x_2]$ be binary forms, with $\deg L_i = 1$ and $\deg Q = 2$, such that L_1, L_2 are non-proportional and Q is irreducible over \mathbb{Q} . Let $\mathcal{B} \subseteq [-1, 1]^2$ be a convex region whose boundary is defined by a piecewise continuously differentiable function. Assume that $L_i(\mathbf{x}) > 0$ and $Q(\mathbf{x}) > 0$ for every $\mathbf{x} \in \mathcal{B}$. The workhorse in this paper is an asymptotic formula for sums akin to

$$\sum_{\substack{\mathbf{x} \in \mathbb{Z}^2 \cap X\mathcal{B} \\ \gcd(x_1, x_2)=1}} \frac{\tau(L_1(\mathbf{x})L_2(\mathbf{x})Q(\mathbf{x}))}{\max\{|x_1|, |x_2|\}^2},$$

where τ is the divisor function and $X\mathcal{B} := \{X\mathbf{x} : \mathbf{x} \in \mathcal{B}\}$.

In fact the arguments that appear in our work call for a rather more general type of sum. Suppose that $g = h * \tau$ is the Dirichlet convolution of the divisor function with a multiplicative arithmetic function h that satisfies

$$\sum_{d \in \mathbb{N}} \frac{|h(d)|}{d^{1/4}} \ll 1. \quad (2.1)$$

Let $V \subseteq [0, 1]^4$ be a region cut out by a finite number of hyperplanes with absolutely bounded coefficients. For any $Y \geq 2$ we define

$$g(L_1(\mathbf{x}), L_2(\mathbf{x}), Q(\mathbf{x}); Y; V) := \sum_{\substack{d | L_1(\mathbf{x})L_2(\mathbf{x})Q(\mathbf{x}) \\ d_i = \gcd(d, L_i(\mathbf{x})), \ d_3 = \gcd(d, Q(\mathbf{x})) \\ (\frac{\log d_1}{\log Y}, \frac{\log d_2}{\log Y}, \frac{\log d_3}{2 \log Y}, \frac{\log \max\{|x_1|, |x_2|\}}{\log Y}) \in V}} (1 * h)(d).$$

Then we will encounter sums of the shape

$$S_g(X, Y; V) := \sum_{\substack{\mathbf{x} \in \mathbb{Z}^2 \cap X\mathcal{B} \\ \gcd(x_1, x_2)=1}} \frac{g(L_1(\mathbf{x}), L_2(\mathbf{x}), Q(\mathbf{x}); Y; V)}{\max\{|x_1|, |x_2|\}^2}.$$

If one takes $V = [0, 1]^4$ and Y a multiple of X then $g(L_1, L_2, Q; Y; V) = g(L_1 L_2 Q)$. Moreover, if one takes

$$h(n) = U(n) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{otherwise,} \end{cases}$$

then one arrives at exactly the sum involving τ that was mentioned above.

For any prime p and $\nu_1, \nu_2, \nu_3 \geq 0$, let

$$\varrho_p^\dagger(\nu_1, \nu_2, \nu_3) := \# \left\{ \mathbf{x} \in (\mathbb{Z}/p^{\nu_1+\nu_2+\nu_3+1}\mathbb{Z})^2 : \begin{array}{l} p \nmid \mathbf{x}, \\ p^{\nu_i} \parallel L_i(\mathbf{x}), \\ p^{\nu_3} \parallel Q(\mathbf{x}) \end{array} \right\} \quad (2.2)$$

and

$$\overline{\varrho}_p^\dagger(\nu_1, \nu_2, \nu_3) := p^{-2(\nu_1+\nu_2+\nu_3+1)} \varrho_p^\dagger(\nu_1, \nu_2, \nu_3). \quad (2.3)$$

Here, we follow the convention that $p^\nu \parallel n$ if and only if $\nu_p(n) = \nu$. The following asymptotic formula is established in our companion paper [5, Corollary 3].

Lemma 5. — *Let $\varepsilon > 0$. Assume that $2 \leq X \leq Y \leq X^{1/\varepsilon}$. Then we have*

$$S_g(X, Y; V) = 4C^* \text{vol}(\mathcal{B}) \text{vol}(V_0) (\log Y)^4 + O_{L_i, Q}((\log X^2/Y)(\log Y)^3 + (\log X)^{3+\varepsilon}),$$

where

$$C^* := \prod_p \left(1 - \frac{1}{p}\right)^3 \sum_{\nu \in \mathbb{Z}_{\geq 0}^3} g(p^{\nu_1 + \nu_2 + \nu_3}) \bar{g}_p^\dagger(\nu_1, \nu_2, \nu_3) \quad (2.4)$$

and

$$V_0 := V \cap \left\{ \mathbf{v} \in [0, 1]^4 : \max\{v_1, v_2, v_3\} \leq v_4 \leq \frac{1}{2} \right\}. \quad (2.5)$$

We will ultimately apply Lemma 5 with $L_1 L_2 Q(a, b)$ equal to the discriminant $\Delta(a, b)$ of the conic (1.6). The exponent of $\log Y$ in the lemma reflects the fact that we are dealing with binary forms with three irreducible factors. The attentive reader will observe a correlation with our description of the rank of $\text{Pic } X$ in §1.

We will be interested in applications of Lemma 5 when $g : \mathbb{N} \rightarrow \mathbb{R}$ is the multiplicative arithmetic function defined via

$$g(p^\nu) := \begin{cases} \max\{1, \nu - 1\}, & \text{if } p = 2, \\ 1 + \nu(1 - 1/p)/(1 + 1/p), & \text{if } p > 2. \end{cases} \quad (2.6)$$

It easily follows that $g = h * \tau$, with

$$h(p^\nu) = (g * \mu * \mu)(p^\nu) = \begin{cases} 0, & \text{if } p > 2 \text{ and } \nu \geq 2, \\ -2/(p+1), & \text{if } p > 2 \text{ and } \nu = 1, \\ 0, & \text{if } p = 2 \text{ and } \nu \geq 4, \\ 1, & \text{if } p = 2 \text{ and } \nu = 3, \\ 0, & \text{if } p = 2 \text{ and } \nu = 2, \\ -1, & \text{if } p = 2 \text{ and } \nu = 1. \end{cases} \quad (2.7)$$

In particular $|h(n)| \ll n^{-1+\varepsilon}$ for any $\varepsilon > 0$, whence (2.1) holds.

3. Preliminary manipulations

Recall the definition of the quadratic forms Φ_1, Φ_2 from (1.2) and (1.4). We begin this section by relating $N_{U, H}(B)$ to the quantity

$$N_1(B) := \# \left\{ \mathbf{x} \in \mathbb{N}^5 : \begin{array}{l} \gcd(x_0, \dots, x_3) = 1, \max\{x_0, \dots, x_3\} \leq B, \\ \Phi_1(\mathbf{x}) = \Phi_2(\mathbf{x}) = 0, \{x_0, x_1\} \neq \{x_2, x_3\} \end{array} \right\}, \quad (3.1)$$

in which the main difference is that the count is restricted to positive integer solutions. This is achieved in the following result.

Lemma 6. — *We have $N_{U, H}(B) = 8N_1(B) + O(B)$.*

Proof. — It follows from the calculation of the lines in §10 that the condition $x \in U$ is equivalent to $\{|x_0|, |x_1|\} \neq \{|x_2|, |x_3|\}$, for any $x = [\mathbf{x}] \in X$. We therefore deduce that

$$N_{U,H}(B) = \frac{1}{2} \# \left\{ \mathbf{x} \in \mathbb{Z}^5 : \begin{array}{l} \gcd(x_0, \dots, x_4) = 1, \|\mathbf{x}\| \leq B, \\ \Phi_1(\mathbf{x}) = \Phi_2(\mathbf{x}) = 0, \{|x_0|, |x_1|\} \neq \{|x_2|, |x_3|\} \end{array} \right\}.$$

It follows from the equation $\Phi_2(\mathbf{x}) = 0$ that the condition $\gcd(x_0, \dots, x_4) = 1$ is equivalent to $\gcd(x_0, x_1, x_2, x_3) = 1$ for any vector \mathbf{x} in which we are interested. Furthermore, for \mathbf{x} such that $\Phi_1(\mathbf{x}) = \Phi_2(\mathbf{x}) = 0$, it is clear from (1.5) that

$$\|\mathbf{x}\| = \max \left\{ |x_0|, |x_1|, |x_2|, |x_3|, \frac{1}{\sqrt{3}} \sqrt{x_0^2 + x_1^2 + x_2^2 - x_3^2} \right\} = \max\{|x_0|, \dots, |x_3|\}.$$

We proceed to consider the contribution from the vectors $\mathbf{x} \in \mathbb{Z}^5$ for which $\|\mathbf{x}\| \leq B$ and

$$\Phi_1(\mathbf{x}) = \Phi_2(\mathbf{x}) = 0, \quad x_0 x_1 x_2 x_3 x_4 = 0.$$

Let us begin with the case $x_i = 0$, for $i \in \{0, 1, 2, 3\}$. This hyperplane section leads us to estimate the number of solutions to an equation of the form $x^2 \pm y^2 \pm 2z^2 = 0$ with $\gcd(x, y, z) = 1$ and $\max\{|x|, |y|, |z|\} \leq B$. An application of Lemma 2 therefore yields a contribution of $O(B)$ from this case. Turning to the contribution from the case $x_4 = 0$ and $x_0 x_1 x_2 x_3 \neq 0$, our task is to estimate the number of primitive vectors $(x, y, z, t) \in \mathbb{Z}_{\neq 0}^4$ such that

$$xy = zt, \quad x^2 + y^2 + z^2 = t^2,$$

with $\max\{|x|, |y|, |z|, |t|\} \leq B$. It is a simple matter to show that there is an overall contribution of $O(B)$ from this case. Bringing everything together we have therefore shown that

$$N_{U,H}(B) = \frac{1}{2} \# \left\{ \mathbf{x} \in \mathbb{Z}_{\neq 0}^5 : \begin{array}{l} \gcd(x_0, \dots, x_3) = 1, \\ \max\{|x_0|, \dots, |x_3|\} \leq B, \\ \Phi_1(\mathbf{x}) = \Phi_2(\mathbf{x}) = 0, \\ \{|x_0|, |x_1|\} \neq \{|x_2|, |x_3|\} \end{array} \right\} + O(B).$$

It remains to show that we can restrict attention to positive values of x_0, \dots, x_4 . But this follows on noting that Φ_2 is invariant under sign changes in the components of \mathbf{x} , and Φ_1 demands that $x_0 x_1$ should share the same sign as $x_2 x_3$. Thus the above cardinality is $16N_1(B)$, with $N_1(B)$ given by (3.1), and the lemma follows. \square

The next stage of the argument involves parametrising the solutions to the equation $\Phi_1(\mathbf{x}) = 0$. It is a simple exercise to show that the set of $x_0, x_1, x_2, x_3 \in \mathbb{N}$ such that $x_0 x_1 = x_2 x_3$ and $\gcd(x_0, \dots, x_3) = 1$, is in bijection with the set of $\mathbf{y} = (y_{02}, y_{03}, y_{12}, y_{13}) \in \mathbb{N}^4$ such that

$$\gcd(y_{02}, y_{13}) = \gcd(y_{03}, y_{12}) = 1,$$

the relation being given by

$$x_0 = y_{02} y_{03}, \quad x_1 = y_{12} y_{13}, \quad x_2 = y_{02} y_{12}, \quad x_3 = y_{03} y_{13}.$$

Define

$$\Psi(\mathbf{y}) := \max\{y_{02} y_{03}, y_{12} y_{13}, y_{02} y_{12}, y_{03} y_{13}\}.$$

On recalling the definition (3.1), it is easy to see that

$$N_1(B) = \# \left\{ (\mathbf{y}, x_4) \in \mathbb{N}^5 : \begin{array}{l} \gcd(y_{02}, y_{13}) = \gcd(y_{03}, y_{12}) = 1, \\ \Phi_2(\mathbf{x}) = 0, \Psi(\mathbf{y}) \leq B, \\ y_{02} \neq y_{13}, y_{03} \neq y_{12}. \end{array} \right\},$$

where \mathbf{x} denotes $(y_{02}y_{03}, y_{12}y_{13}, y_{02}y_{12}, y_{03}y_{13}, x_4)$.

Given integers a, b , recall the definition (1.6) of the plane conic $C_{a,b} \subset \mathbb{P}_{\mathbb{Q}}^2$. Writing $(a, b) = (y_{02}, y_{13})$ and $(x, y, z) = (y_{03}, y_{12}, x_4)$, it follows that $N_1(B)$ is the number of $(a, b, x, y, z) \in \mathbb{N}^5$ such that (1.6) holds, with

$$\gcd(a, b) = \gcd(x, y) = 1, \quad ab, xy \neq 1, \quad \max\{a, b\} \max\{x, y\} \leq B.$$

For $a, b \in \mathbb{N}$ we define the quantities

$$\widetilde{M}_{a,b}(B) := \# \left\{ (x, y, z) \in C_{a,b} \cap \mathbb{N}^3 : \begin{array}{l} \gcd(x, y) = 1, \\ \max\{a, b\} < \max\{x, y\}, \\ \max\{a, b\} \max\{x, y\} \leq B \end{array} \right\}, \quad (3.2)$$

$$\widehat{M}_{a,b}(B) := \# \left\{ (x, y, z) \in C_{a,b} \cap \mathbb{N}^3 : \begin{array}{l} \gcd(x, y) = 1, xy \neq 1 \\ \max\{a, b\} \max\{x, y\} \leq B \end{array} \right\}, \quad (3.3)$$

$$M_{a,b}(B) := \# \left\{ (x, y, z) \in C_{a,b} \cap \mathbb{N}^3 : \begin{array}{l} \gcd(x, y) = 1, \\ \max\{a, b\} \max\{x, y\} \leq B \end{array} \right\}, \quad (3.4)$$

where $C_{a,b} \subset \mathbb{P}_{\mathbb{Q}}^2$ is the conic (1.6). There is an abuse of notation here, in that we are interested in vectors $(x, y, z) \in \mathbb{N}^3$ that lie in the affine cone above $C_{a,b}$, rather than points $[x, y, z] \in C_{a,b}(\mathbb{Q})$ in which coordinates are chosen so that $x, y, z > 0$. Note that we must automatically have $xy \neq 1$ in the definition of $\widetilde{M}_{a,b}(B)$ since $1 \leq \max\{a, b\} < \max\{x, y\}$. Hence $\widetilde{M}_{a,b}(B) \leq \widehat{M}_{a,b}(B) \leq M_{a,b}(B)$. We have

$$N_1(B) = \sum_{\substack{a, b \leq B \\ \gcd(a, b) = 1, ab \neq 1}} \widehat{M}_{a,b}(B)$$

and we are now ready to establish the following result.

Lemma 7. — *We have*

$$N_1(B) = 2 \sum_{\substack{a, b < \sqrt{B} \\ \gcd(a, b) = 1, ab \neq 1}} \widetilde{M}_{a,b}(B) + O(B(\log B)^3),$$

where $\widetilde{M}_{a,b}(B)$ is given by (3.2).

Proof. — Fundamental to our argument is the observation $(x, y, z) \in C_{a,b} \cap \mathbb{N}^3$ if and only if $(b, a, z) \in C_{y,x} \cap \mathbb{N}^3$. From this it easily follows that

$$N_1(B) = 2 \sum_{\substack{a, b < \sqrt{B} \\ \gcd(a, b) = 1, ab \neq 1}} \widetilde{M}_{a,b}(B) + N'_1(B),$$

where

$$N'_1(B) := \sum_{\substack{a, b < \sqrt{B} \\ \gcd(a, b) = 1, ab \neq 1}} \# \left\{ (x, y, z) \in C_{a, b} \cap \mathbb{N}^3 : \begin{array}{l} \gcd(x, y) = 1, \\ \max\{a, b\} = \max\{x, y\} \end{array} \right\}.$$

To establish the lemma it therefore suffices to show that $N'_1(B) \ll B(\log B)^3$. Without loss of generality we may view $N'_1(B)$ as being the overall contribution from vectors such that

$$\max\{a, b\} = b = y = \max\{x, y\},$$

the remaining 3 cases being handled in an identical fashion. Arguing as above, we find that

$$N'_1(B) \leq 4 \# \left\{ (a, x, y, z) \in \mathbb{N}^4 : \begin{array}{l} y^4 + x^2 a^2 + y^2(a^2 - x^2) = 2z^2, \\ \gcd(y, xa) = 1, \\ \max\{x, a\} \leq y \leq \sqrt{B} \end{array} \right\}.$$

The overall contribution to the right hand side from the case $a = y$, so that $a = y = 1$, is clearly $O(1)$. We will estimate the remaining contribution by first fixing a choice of y and a , and then summing over all available x and z . First we note that y is necessarily odd in any 4-tuple (a, x, y, z) .

For fixed a, y the form $x^2(y^2 - a^2) + 2z^2$ defines a positive definite binary quadratic form of discriminant $D = -8(y^2 - a^2) \neq 0$. In particular the total number of available x, z is bounded by the number of representations $R_D(y^2(y^2 + a^2))$ of $y^2(y^2 + a^2)$ by a complete system of inequivalent forms of discriminant D . It easily follows that

$$N'_1(B) \leq 4 \sum_{\substack{y, a \in \mathbb{N} \\ a < y \leq \sqrt{B} \\ 2 \nmid y, \gcd(a, y) = 1}} R_D(y^2(y^2 + a^2)) + O(1),$$

with $D = -8(y^2 - a^2)$. By the classical theory of binary quadratic forms we have

$$R_D(y^2(y^2 + a^2)) = 2 \sum_{d|y^2(y^2 + a^2)} \chi_D(d) \ll \tau(y^2(y^2 + a^2)) = \tau(y^2)\tau(y^2 + a^2),$$

where χ_D is the Kronecker symbol. The last equality follows since $\gcd(a, y) = 1$. An application of Lemma 3 therefore reveals that there is an absolute constant $c > 0$ such that

$$\begin{aligned} N'_1(B) &\ll \sum_{y \leq \sqrt{B}} \tau(y^2) \sum_{a \leq \sqrt{B}} \tau(y^2 + a^2) + \sqrt{B} \ll \sqrt{B} \log B \sum_{y \leq \sqrt{B}} \varphi^\dagger(y)^c \tau(y^2) \\ &\ll B(\log B)^3, \end{aligned}$$

where φ^\dagger is given by (1.10). This completes the proof of the lemma. \square

4. Reducing the range of summation

It will greatly facilitate our arguments if we can reduce the range of summation for a, b in the statement of Lemma 7. The main aim of this section is to establish the following result.

Lemma 8. — *Let $T > 0$ and recall the definition (3.4) of $M_{a,b}(B)$. Then we have the following estimates*

$$\begin{aligned} \sum_{\substack{\sqrt{B}/T < \max\{a,b\} \leq \sqrt{B} \\ \gcd(a,b)=1, ab \neq 1}} M_{a,b}(B) &\ll B(\log B)^3 \log(T+2) \\ \sum_{\substack{T \min\{a,b, |a-b|\} < \max\{a,b\} \leq \sqrt{B} \\ \gcd(a,b)=1, ab \neq 1}} M_{a,b}(B) &\ll \frac{B(\log B)^4}{T^{2/3}} + B(\log B)^3. \end{aligned}$$

Applying the second estimate in Lemma 8 with $T = 1/2$ and inserting this into Lemmas 6 and 7, we easily conclude that

$$N_{U,H}(B) \ll B(\log B)^4.$$

Achieving a corresponding lower bound for $N_{U,H}(B)$ is straightforward. We will have to work much harder to deduce an asymptotic formula.

Proof of Lemma 8. — The main tool in our proof of this result is Lemma 2. For the conic $C_{a,b}$ defined in (1.6) we have $\Delta = -2(a^4 - b^4)$ and $\Delta_0 \ll 1$, since $\gcd(a, b) = 1$ by assumption. It therefore follows that

$$M_{a,b}(B) \ll \tau(|a^4 - b^4|) \left(1 + \frac{B}{\max\{a, b\}^{5/3} |a - b|^{1/3}} \right). \quad (4.1)$$

Thus there are two terms to consider, both of which we must sum over the relevant values of a, b . Lemma 4 immediately implies that the first term contributes $O(B(\log B)^3)$ to the two sums in the lemma, thereby allowing us to concentrate on the contribution from the second term. We will show that

$$\sum_{\max\{a, x/y\} < b \leq x} \frac{\tau(|a^4 - b^4|)}{b^{5/3} |a - b|^{1/3}} \ll (\log x)^3 \log(y+2) \quad (4.2)$$

and

$$\sum_{0 \leq y \min\{a, b-a\} < b \leq x} \frac{\tau(|a^4 - b^4|)}{b^{5/3} |a - b|^{1/3}} \ll \frac{(\log x)^4}{y^{2/3}} + 1, \quad (4.3)$$

for any $y > 0$. Combining these with (4.1), one easily verifies that these estimates are enough to complete the proof of the lemma.

It therefore remains to establish (4.2) and (4.3). Beginning with the estimation of the sum in (4.2), which we denote by $S(x, y)$, it will be convenient to set $u = b + a$ and $v = b - a$. In particular $b^4 - a^4 = uv(u^2 + v^2)/2$. We deduce that

$$S(x, y) \ll \sum_{\substack{x/y < u \leq 2x \\ v < u}} \frac{\tau(uv(u^2 + v^2))}{u^{5/3} v^{1/3}} \ll \sum_{x/y < u \leq 2x} \frac{\tau(u)}{u^{5/3}} \sum_{v < u} \frac{\tau(v(u^2 + v^2))}{v^{1/3}}.$$

But then it follows from combining Lemma 3 with partial summation that

$$S(x, y) \ll (\log x)^2 \sum_{x/y < u \leq 2x} \frac{\tau(u)}{u} \varphi^\dagger(u)^c \ll (\log x)^3 \log(y+2),$$

for an absolute constant $c > 0$. This completes the proof of (4.2).

It remains to estimate the sum in (4.3), which we also denote by $S(x, y)$. It easily follows that

$$S(x, y) \ll \sum_{ya < b \leq x} \frac{\tau(|a^4 - b^4|)}{b^{5/3}a^{1/3}} + \sum_{0 \leq y(b-a) < b \leq x} \frac{\tau(|a^4 - b^4|)}{b^{5/3}(b-a)^{1/3}}.$$

On taking dyadic intervals for $a < b/y$ and $b \leq x$, we see that the first sum here is

$$\leq \sum_{A, A'} \frac{1}{A^{5/3}A'^{1/3}} \sum_{\substack{A \leq b < 2A \\ A' \leq a < 2A'}} \tau(|a^4 - b^4|) \ll 1 + \sum_{A, A'} \frac{A'^{2/3}(\log x)^3}{A^{2/3}} \ll \frac{(\log x)^4}{y^{2/3}} + 1,$$

by Lemma 4. Similarly, on setting $u = b + a$ and $v = b - a$ as above and taking dyadic intervals for $u \leq 2x$ and $v \leq u/y$, we deduce from Lemma 4 that the remaining sum is

$$\ll \sum_{A, A'} \frac{1}{A^{5/3}A'^{1/3}} \sum_{u \leq A, v \leq A'} \tau(uv(u^2 + v^2)) \ll \frac{(\log x)^4}{y^{2/3}} + 1,$$

as required for (4.3). This completes the proof of the lemma. \square

The sort of argument used to prove (4.2) and (4.3) will feature quite often in our work. Recall the definitions (1.11) of Z_1 and Z_2 . It is easy to see that $\log Z_1 = (\log B)/(\log \log B)$ and 2^{Z_2} has order of magnitude $\log B$. Our next task is twofold. Firstly, we would like to be able to restrict attention to values of a, b belonging to the set

$$\mathcal{A}_1 := \left\{ (a, b) \in \mathbb{N}^2 : \begin{array}{l} \max\{a, b\} < \sqrt{B}/Z_1^c, \\ \max\{a, b\} \leq Z_2^2 \min\{a, b, |a - b|\}, \\ \gcd(a, b) = 1, ab \neq 1, \\ \nu_2(a^2 - b^2) \leq Z_2 \end{array} \right\}. \quad (4.4)$$

Here $c > 0$ is an absolute constant that when appearing as an exponent of Z_1 , we will view as being sufficiently large to ensure that all of our error terms are satisfactory. Secondly, we wish to show that $\widetilde{M}_{a,b}(B)$ can be replaced by $M_{a,b}(B)$ in Lemma 7, with a negligible error. Both of these objectives are achieved in the following result.

Lemma 9. — *We have*

$$N_1(B) = 2 \sum_{(a,b) \in \mathcal{A}_1} M_{a,b}(B) + O\left(\frac{B(\log B)^4}{\log \log B}\right),$$

where $M_{a,b}(B)$ is given by (3.4) and \mathcal{A}_1 is given by (4.4).

Proof. — Taken together with Lemma 7, an application of Lemma 8 easily implies that

$$N_1(B) = 2 \sum_{\substack{\max\{a,b\} < \sqrt{B}/Z_1^c \\ \max\{a,b\} \leq Z_2^2 \min\{a,b,|a-b|\} \\ \gcd(a,b)=1, ab \neq 1}} \widetilde{M}_{a,b}(B) + O\left(\frac{B(\log B)^4}{\log \log B}\right). \quad (4.5)$$

Let us proceed by indicating how to replace $\widetilde{M}_{a,b}(B)$ by $M_{a,b}(B)$ in the summand. Using the fact that $\max\{a, b\} < \sqrt{B}/Z_1^c$ it is easy to see that

$$M_{a,b}(B) - \widetilde{M}_{a,b}(B) \leq \# \left\{ (x, y, z) \in C_{a,b} \cap \mathbb{N}^3 : \begin{array}{l} \gcd(x, y) = 1, \\ \max\{x, y\} < \sqrt{B}/Z_1^c \end{array} \right\}.$$

But then it follows that we are interested in values of x, y such that

$$\max\{x, y\} \leq \frac{B}{Z_1^{2c} \max\{a, b\}}.$$

An argument based on the proof of Lemma 8 reveals that we can replace $\widetilde{M}_{a,b}(B)$ by $M_{a,b}(B)$, as required.

In order to restrict to a summation over $(a, b) \in \mathcal{A}_1$ we must consider the contribution from a, b such that $\nu_2(a^2 - b^2) > Z_2$. In view of Lemma 4 and (4.1), one sees that the overall contribution is bounded by

$$\begin{aligned} & \ll \sum_{\substack{a, b < \sqrt{B}/Z_1^c \\ \gcd(a, b) = 1, ab \neq 1 \\ \nu_2(a^2 - b^2) > Z_2}} \tau(|a^4 - b^4|) \left(1 + \frac{B}{(a+b)^{5/3} |a-b|^{1/3}} \right) \\ & \ll \frac{B(\log B)^3}{Z_1^{2c}} + \frac{B}{Z_2} \sum_{\substack{a, b < \sqrt{B}/Z_1^c \\ \gcd(a, b) = 1, ab \neq 1}} \frac{\nu_2(a^2 - b^2) \tau(|a^4 - b^4|)}{(a+b)^{5/3} |a-b|^{1/3}}. \end{aligned}$$

Let us suppose without loss of generality that $a > b$ in the remaining sum. On writing $(u, v) = (a+b, a-b)$, we see that the sum over a, b is

$$\ll \sum_{v < u < 2\sqrt{B}/Z_1^c} \frac{\nu_2(uv) \tau(uv(u^2 + v^2))}{u^{5/3} v^{1/3}} \ll \sum_{v < u < 2\sqrt{B}/Z_1^c} \frac{\tau''(uv(u^2 + v^2))}{u^{5/3} v^{1/3}},$$

where τ'' is defined via

$$\tau''(p^\nu) := \begin{cases} \nu + 1, & \text{if } p > 2, \\ (\nu + 1)^2, & \text{if } p = 2. \end{cases}$$

One easily confirms that the proof of Lemma 3 goes through with τ replaced by τ'' , whence the argument used to establish (4.2) applies in this setting too. Thus it follows that the contribution to the main term in (4.5) from $\nu_2(a^2 - b^2) > Z_2$ is satisfactory. This completes the proof of the lemma. \square

5. Parametrisation of the conics

In this section we concern ourselves with estimating $M_{a,b}(B)$, as given by (3.4). Recall the definition (1.6) of $C_{a,b}$. Fundamental to our approach is the observation that for each $a, b \in \mathbb{N}$, the conic $C_{a,b}$ always contains the rational point ξ in (1.7). We are therefore in a position to parametrise all of the rational points on the conic by

considering the residual intersection with $C_{a,b}$ of an arbitrary line through ξ . Define the binary quadratic forms

$$\begin{aligned} Q_1(s, t) &:= 2s^2 + (a^2 - b^2)t^2 - 4ast, \\ Q_2(s, t) &:= -2s^2 + (a^2 - b^2)t^2, \\ Q_3(s, t) &:= -2as^2 + 2(a^2 - b^2)st - a(a^2 - b^2)t^2, \end{aligned} \tag{5.1}$$

for given $a, b \in \mathbb{N}$ such that $\gcd(a, b) = 1$ and $ab \neq 1$. In particular we will always have $a^2 \pm b^2 \neq 0$. One easily checks that

$$Q_3(s, t) = t^{-1}(sQ_1(s, t) + (s - at)Q_2(s, t)). \tag{5.2}$$

The main aim of this section is to establish the following result.

Lemma 10. — *We have*

$$M_{a,b}(B) = \# \left\{ (s, t) \in \mathbb{Z}^2 : \begin{array}{l} \gcd(s, t) = 1, \ s(s - at) \neq 0, \\ s/t \neq (a^2 - b^2)/(2a), \\ t > 0, \ 0 < -Q_3(s, t), \\ 0 < -Q_j(s, t) \leq \frac{\lambda B}{\max\{a, b\}} \text{ for } j = 1, 2 \end{array} \right\} + O(1),$$

where $\lambda = \gcd(Q_1(s, t), Q_2(s, t))$.

Proof. — Recall the definitions (1.6) and (1.7) of $C_{a,b}$ and ξ , respectively. Let

$$L_\xi := \{(a^2 - b^2)x - (a^2 + b^2)y - 2az = 0\}.$$

This is the tangent line to $C_{a,b}$ along ξ . Let \mathcal{L} denote the set of projective lines in $\mathbb{P}_{\mathbb{Q}}^2$ that pass through ξ , and let $\mathcal{L}(\mathbb{Q})$ be the corresponding subset that are defined over \mathbb{Q} . We will write $\mathcal{U} = \mathcal{L} \setminus \{L_\xi\}$. Now let $U \subset C_{a,b}$ denote the open subset formed by deleting ξ from the conic. As is well-known, the sets $U(\mathbb{Q})$ and $\mathcal{U}(\mathbb{Q})$ are in bijection.

The general element of $\mathcal{L}(\mathbb{Q})$ is given by

$$L_{s,t} := \{sx + (s - at)y - tz = 0\},$$

for $s, t \in \mathbb{Z}$ such that $\gcd(s, s - at, t) = \gcd(s, t) = 1$. There is clearly a bijection between those lines for which $t < 0$ and those for which $t > 0$. We henceforth fix our attention on the latter. In order to have a point in $\mathcal{U}(\mathbb{Q})$ we must insist that $(s, t) \neq (a^2 - b^2, 2a)$, which is clearly equivalent to the condition $s/t \neq (a^2 - b^2)/(2a)$. Moreover, there are only $O(1)$ points in $\mathcal{U}(\mathbb{Q})$ that correspond to taking $t = 0$.

To make the bijection between $U(\mathbb{Q})$ and $\mathcal{U}(\mathbb{Q})$ completely explicit a routine calculation reveals that a point $[x, y, z]$ is in the intersection $L_{s,t} \cap C_{a,b}$ if and only if $x = -y$, or else $Q_2(s, t)x = Q_1(s, t)y$, in the notation of (5.1). The first alternative leads us to the point ξ , which is to be ignored. The second alternative implies that

$$tQ_1(s, t)z = (sQ_1(s, t) + (s - at)Q_2(s, t))x = tQ_3(s, t)x,$$

by (5.2), in the equation for $L_{s,t}$. Any vector $(x, y, z) \in \mathbb{Z}^3$ that represents a point in $C_{a,b}(\mathbb{Q})$ is primitive if and only if $\gcd(x, y) = 1$. Writing λ as in the statement of the lemma, we therefore find that

$$(x, y, z) = \pm(Q_1(s, t)/\lambda, Q_2(s, t)/\lambda, Q_3(s, t)/\lambda).$$

for any $[x, y, z] \in (U \cap L_{s,t})(\mathbb{Q})$.

So far we have recorded an explicit bijection between elements of $U(\mathbb{Q})$ and $\mathcal{W}(\mathbb{Q})$. In deriving an expression for $M_{a,b}(B)$ in terms of this bijection we will need to restrict the corresponding values of s, t that are to be considered. Specifically we will only be interested in the values of $(s, t) \in \mathbb{Z} \times \mathbb{Z}_{>0}$ for which $\gcd(s, t) = 1$ and the corresponding values of x, y, z lie in the region defined by the inequalities

$$x, y, z > 0, \quad \max\{a, b\} \max\{x, y\} \leq B.$$

Finally, we will want to exclude the possibility that $s(s - at) = 0$. Since there are only $O(1)$ such values of s, t to worry about, this therefore concludes the proof that

$$M_{a,b}(B) = \sum_{\varepsilon \in \{\pm 1\}} \# \left\{ (s, t) \in \mathbb{Z}^2 : \begin{array}{l} \gcd(s, t) = 1, \quad s(s - at) \neq 0, \\ s/t \neq (a^2 - b^2)/(2a), \\ t > 0, \quad 0 < \varepsilon Q_3(s, t), \\ 0 < \varepsilon Q_j(s, t) \leq \frac{\lambda B}{\max\{a, b\}} \text{ for } j = 1, 2 \end{array} \right\} + O(1).$$

Our final task is to show that $\varepsilon = -1$ in this expression. Suppose that $\varepsilon = +1$. The height conditions imply that $Q_2(s, t) > 0$ and $t > 0$. But then it follows that we must necessarily have $\text{sgn}(a - b) = 1$. However we must also have $Q_3(s, t) > 0$, whence

$$-\sqrt{a^2 - b^2}(s - t')^2 - (2a - \sqrt{a^2 - b^2})s^2 - (a - \sqrt{a^2 - b^2})t'^2 > 0,$$

where we have written $t' = t\sqrt{a^2 - b^2}$ for ease of notation. This contradiction establishes the lemma. \square

Lemma 10 allows us to translate the underlying problem to one that involves counting primitive lattice points in a complicated region contained in \mathbb{R}^2 . Before we proceed to consider this region in more detail, it will be necessary to gain a better understanding of the greatest common divisor λ .

Lemma 11. — *Let a, b be coprime positive integers such that $ab \neq 1$. Let s, t be coprime integers and let $\lambda = \gcd(Q_1(s, t), Q_2(s, t))$. Then we have $\lambda = 2^\nu \lambda_1 \lambda_2$, where*

$$\nu = \begin{cases} 0, & \text{if } 2 \mid ab \text{ and } 2 \nmid t, \\ 1, & \text{if } 2 \mid ab \text{ and } 2 \mid t, \\ 1, & \text{if } 2 \nmid ab \text{ and } 2 \nmid s, \\ \min\{2 + \nu_2(s), \nu_2(a^2 - b^2)\}, & \text{if } 2 \nmid ab \text{ and } 2 \mid s, \end{cases} \quad (5.3)$$

and

$$\lambda_1 = (s, a^2 - b^2)_b, \quad \lambda_2 = (s - at, a^2 + b^2)_b,$$

in the notation of (1.9).

Proof. — Let us write $\lambda = 2^\nu \lambda'$ for $\nu \geq 0$ and $\lambda' \in \mathbb{N}$ odd. Observe that

$$\lambda = \gcd(Q_1, Q_2) = \gcd(Q_1 - Q_2, Q_2) = \gcd(4s(s - at), -2s^2 + (a^2 - b^2)t^2). \quad (5.4)$$

The precise value of ν will depend intimately on the 2-adic valuations of a, b, s and t . Let us write

$$a = 2^\alpha a', \quad b = 2^\beta b', \quad s = 2^\sigma s', \quad t = 2^\tau t',$$

for $\alpha, \beta, \sigma, \tau \geq 0$ and $a'b's't'$ odd. It follows from (5.4) that

$$\nu = \min\{2 + \sigma + \nu_2(2^\sigma s' - 2^{\alpha+\tau} t' a'), \nu_2(-2^{1+2\sigma} s'^2 + (2^{2\alpha} a'^2 - 2^{2\beta} b'^2) 2^{2\tau} t'^2)\}.$$

Suppose first that $\alpha = \beta = 0$. Then

$$\nu = \min\{2 + \sigma + \nu_2(2^\sigma s' - 2^\tau t' a'), \nu_2(-2^{1+2\sigma} s'^2 + (a'^2 - b'^2) 2^{2\tau} t'^2)\}.$$

Thus either $\sigma = 0$, in which case $\nu = 1$, or else $\sigma \geq 1$. In the latter case $\tau = 0$ and it follows that

$$\nu = \min\{2 + \sigma, \nu_2(-2^{1+2\sigma} s'^2 + (a'^2 - b'^2) t'^2)\} = \min\{2 + \sigma, \nu_2(a'^2 - b'^2)\}.$$

Differentiating according to whether $\alpha \geq 1$ and $\beta = 0$, or $\alpha = 0$ and $\beta \geq 1$, it is easily checked that

$$\nu = \begin{cases} 1, & \text{if } \tau \geq 1, \\ 0, & \text{if } \tau = 0. \end{cases}$$

Turning to the odd part λ' of λ , we deduce from (5.4) that

$$\begin{aligned} \lambda' &= (s(s - at), -2s^2 + (a^2 - b^2)t^2)_b \\ &= (s, -2s^2 + (a^2 - b^2)t^2)_b (s - at, -2s^2 + (a^2 - b^2)t^2)_b \\ &= (s, a^2 - b^2)_b (s - at, a^2 + b^2)_b. \end{aligned}$$

This completes the proof of the lemma. \square

Note that the last equality in (5.3) is only possible if $\nu \geq 2$ since $a^2 - b^2$ is divisible by 4 if a and b are both odd. Define the three quadratic forms

$$\begin{aligned} p_u(s, t) &:= -2s^2 - \frac{1 - u^2}{|1 - u^2|} t^2 + \frac{4}{\sqrt{|1 - u^2|}} st, \\ q_u(s, t) &:= 2s^2 - \frac{1 - u^2}{|1 - u^2|} t^2, \\ r_u(s, t) &:= 2s^2 - 2 \frac{1 - u^2}{\sqrt{|1 - u^2|}} st + \frac{1 - u^2}{|1 - u^2|} t^2, \end{aligned} \tag{5.5}$$

for any positive $u \neq 1$. We may then write

$$Q_1(s, t) = -p_{b/a}(s, \alpha t), \quad Q_2(s, t) = -q_{b/a}(s, \alpha t), \quad Q_3(s, t) = -ar_{b/a}(s, \alpha t),$$

in (5.1), where $\alpha = \sqrt{|a^2 - b^2|}$.

For any $X > 0$ we define the region

$$\begin{aligned} \mathcal{R}(X) &:= \left\{ (s, t) \in \mathbb{R} \times \mathbb{R}_{>0} : \begin{array}{l} Q_3(s, t) < 0, \\ 0 > Q_1(s, t), Q_2(s, t) \geq -X \end{array} \right\} \\ &= \left\{ (s, t) \in \mathbb{R} \times \mathbb{R}_{>0} : \begin{array}{l} 0 < r_{b/a}(s, \sqrt{|a^2 - b^2|} t), \\ 0 < p_{b/a}(s, \sqrt{|a^2 - b^2|} t) \leq X, \\ 0 < q_{b/a}(s, \sqrt{|a^2 - b^2|} t) \leq X \end{array} \right\}. \end{aligned} \tag{5.6}$$

Furthermore, we set

$$\mathcal{R}'(X) := \{(s, t) \in \mathcal{R}(X) : s(s - at) \neq 0, s/t \neq (a^2 - b^2)/(2a)\} \tag{5.7}$$

and

$$\mathcal{R}^\dagger(X) := \left\{ (s, t) \in \mathbb{R}^2 : \begin{array}{l} Q_j(s, t) \neq 0 \text{ for } 1 \leq j \leq 3, \\ |Q_1(s, t)|, |Q_2(s, t)| \leq X \\ st(s - at) \neq 0, s/t \neq (a^2 - b^2)/(2a) \end{array} \right\}. \quad (5.8)$$

Bringing together Lemmas 10 and 11, we may deduce that

$$M_{a,b}(B) = \sum_{\nu=0}^{\infty} \sum_{\lambda_1 | a^2 - b^2}^b \sum_{\lambda_2 | a^2 + b^2}^b L(a, b; B; \nu, \lambda_1, \lambda_2) + O(1), \quad (5.9)$$

where $L(B) = L(a, b; B; \nu, \lambda_1, \lambda_2)$ is the number of $(s, t) \in \mathbb{Z}^2$ subject to the following conditions:

1. $\gcd(s, t) = 1$;
2. $(s, t) \in \mathcal{R}(2^\nu \lambda_1 \lambda_2 B / \max\{a, b\})$;
3. $s/t \neq (a^2 - b^2)/(2a)$ and $s(s - at) \neq 0$;
4. $\lambda_1 \mid s$ and $\lambda_2 \mid s - at$;
5. $(s/\lambda_1, (a^2 - b^2)/\lambda_1)_b = 1$ and $((s - at)/\lambda_2, (a^2 + b^2)/\lambda_2)_b = 1$; and
6. the 2-adic orders of s, t are determined by ν via (5.3).

Here we recall the convention that the symbol \sum^b implies a restriction to odd parameters in the summation and we note that (2) and (3) are together equivalent to $(s, t) \in \mathcal{R}'(2^\nu \lambda_1 \lambda_2 B / \max\{a, b\})$ in the notation of (5.7).

6. Removing the coprimality conditions

The way forward should now be clear. For given values of a, b , and appropriate values of ν, λ_1 and λ_2 , we must attempt to produce an asymptotic formula for the number of primitive lattice points in a complicated region in \mathbb{R}^2 . We will do so using exponential sums. The first step, however, is to remove the coprimality conditions that go into the definition of $L(B)$. Let $\mathbf{k} = (k_1, k_2)$ and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ such that $\gcd(k_1 k_2 \lambda_1 \lambda_2, ab) = 1$. Define the set

$$\Lambda(\mathbf{k}, \boldsymbol{\lambda}, \ell) := \{(s, t) \in \mathbb{Z}^2 : [k_1 \lambda_1, \ell] \mid s, k_2 \lambda_2 \mid s - at \text{ and } \ell \mid t\}. \quad (6.1)$$

Then $\Lambda(\mathbf{k}, \boldsymbol{\lambda}, \ell) \subseteq \mathbb{Z}^2$ is an integer sublattice of rank 2, whose determinant is given by the following result.

Lemma 12. — *We have*

$$\det \Lambda(\mathbf{k}, \boldsymbol{\lambda}, \ell) = \frac{k_1 k_2 \lambda_1 \lambda_2 \ell^2}{\gcd(k_1 k_2 \lambda_1 \lambda_2, \ell)}.$$

Proof. — Writing $s = \ell s'$ and $t = \ell t'$ it follows that $\det \Lambda = \ell^2 \det \Lambda'$, where Λ' is the set of $(s', t') \in \mathbb{Z}^2$ such that $\lambda'_1 \mid s'$ and $\lambda'_2 \mid s' - t'a$, with λ'_i equal to $k_i \lambda_i / \gcd(k_i \lambda_i, \ell)$. The proof of the lemma then follows on noting that $\gcd(\lambda'_1 \lambda'_2, a) = 1$, since a and b are coprime. \square

We proceed to remove the coprimality conditions that appear in (5). Thus an application of Möbius inversion yields

$$L(B) = \sum_{k_1 | (a^2 - b^2)/\lambda_1}^b \sum_{k_2 | (a^2 + b^2)/\lambda_2}^b \mu(k_1)\mu(k_2)L_{k_1, k_2}(B), \quad (6.2)$$

where now $L_{k_1, k_2}(B)$ is the number of $(s, t) \in \Lambda(\mathbf{k}, \boldsymbol{\lambda}, 1)$ such that (1)–(3) and (6) hold in the definition of $L(B)$.

We now consider the regions (5.6) and (5.7) in more detail. Define

$$\mathcal{S}_u := \{(s, t) \in \mathbb{R} \times \mathbb{R}_{>0} : 0 < p_u(s, t), q_u(s, t) \leq 1, r_u(s, t) > 0\} \quad (6.3)$$

and

$$\mathcal{S}'_u := \left\{ (s, t) \in \mathcal{S}_u : s(s - t/\sqrt{|1 - u^2|}) \neq 0, s/t \neq (1 - u^2)/(2\sqrt{|1 - u^2|}) \right\}, \quad (6.4)$$

for any positive $u \neq 1$. We observe that $(s, t) \in \mathcal{R}'(X)$ if and only if

$$\left(\frac{s}{\sqrt{X}}, \frac{\sqrt{|a^2 - b^2|}t}{\sqrt{X}} \right) \in \mathcal{S}'_{b/a}.$$

We then have the following result.

Lemma 13. — *Let $X > 0$ and $u \neq 1$ be positive. Then we have*

$$\text{vol}(\mathcal{R}'(X)) = \text{vol}(\mathcal{R}(X)) = \frac{X \text{vol}(\mathcal{S}_{b/a})}{\sqrt{|a^2 - b^2|}}.$$

Furthermore, we have

$$\mathcal{R}(X) \subseteq [-c\sqrt{X}, c\sqrt{X}] \times (0, c\sqrt{X/|a^2 - b^2|}], \quad \mathcal{S}_u \subseteq [-c, c] \times (0, c],$$

for an absolute constant $c > 0$.

Proof. — The first part of the lemma is self-evident, and so it remains to establish the bounds on $\mathcal{R}(X)$ and \mathcal{S}_u in the second part. For this it will clearly suffice to show that $s \ll 1$ and $0 < t \ll 1$ for any $(s, t) \in \mathcal{S}_u$. Suppose first that $u^2 > 1$. Then it follows from the inequality $q_u(s, t) \leq 1$ that $2s^2 + t^2 \leq 1$, whence $s, t \ll 1$ in this case. If $u^2 < 1$ then we have $0 < 2s^2 - t^2 \leq 1$ and $0 < -2s^2 - t^2 + 4st/\sqrt{1 - u^2} \leq 1$. In particular it follows that $0 < t < \sqrt{2}s$. Let $\eta > 0$. If $t < s\sqrt{2 - \eta}$ then $1 \geq 2s^2 - t^2 > \eta s^2$. Thus $s \leq 1/\sqrt{\eta}$ and $t \leq \sqrt{2/\eta}$ in this case. Alternatively, if $s\sqrt{2 - \eta} \leq t < \sqrt{2}s$, then we deduce that

$$\begin{aligned} 1 &\geq -2s^2 - t^2 + \frac{4st}{\sqrt{1 - u^2}} = s^2 \left(-2 - \left(\frac{t}{s} \right)^2 + \frac{4}{\sqrt{1 - u^2}} \left(\frac{t}{s} \right) \right) \\ &> 4s^2(-1 + \sqrt{2 - \eta}), \end{aligned}$$

since $1/\sqrt{1 - u^2} > 1$. Taking $\eta = 1/2$ it therefore follows that $s, t \ll 1$ in every case, which completes the proof of the lemma. \square

Recall the definition (4.4) of the set \mathcal{A}_1 . For any $(a, b) \in \mathcal{A}_1$ it follows from (5.3) that we are only interested in values of

$$\nu \leq \max\{1, \nu_2(a^2 - b^2)\} \leq Z_2 = \log \log B.$$

Returning to our expression (6.2) for $L(B)$, we may now show that there is a negligible contribution from large values of k_1, k_2 . Let

$$K := (\log B)^{50}. \quad (6.5)$$

Then we have the following result.

Lemma 14. — *We have*

$$\sum_{(a,b) \in \mathcal{A}_1} \sum_{\nu \leq Z_2} \sum_{k_1 \lambda_1 | a^2 - b^2}^b \sum_{k_2 \lambda_2 | a^2 + b^2}^b L_{k_1, k_2}(B) \ll B,$$

where the summations are subject to $\max\{k_1, k_2\} > K$.

Proof. — Lemma 12 reveals that $\Lambda(\mathbf{k}, \boldsymbol{\lambda}, 1)$ is an integer lattice of rank 2 and determinant $\lambda_1 \lambda_2 k_1 k_2$. Hence it follows from Lemmas 1 and 13 that

$$L_{k_1, k_2}(B) \ll 1 + \frac{\text{vol } \mathcal{R}(X)}{\lambda_1 \lambda_2 k_1 k_2} \ll 1 + \frac{2^\nu B}{\sqrt{|a^2 - b^2|} \max\{a, b\} k_1 k_2},$$

with

$$X = \frac{2^\nu \lambda_1 \lambda_2 B}{\max\{a, b\}}. \quad (6.6)$$

We must now estimate the overall contribution from each of the two terms.

The contribution from the first term is bounded by

$$\begin{aligned} \sum_{a, b < \sqrt{B}/Z_1^c} \sum_{\nu \leq Z_2} \sum_{k_1 \lambda_1 | a^2 - b^2}^b \sum_{k_2 \lambda_2 | a^2 + b^2}^b 1 &\ll \log \log B \sum_{a, b} \tau_3(|a^4 - b^4|) \\ &\ll \frac{B(\log B)^6 \log \log B}{Z_1^{2c}}, \end{aligned}$$

using Lemma 4. Since we are interested in values of $k_1 k_2 \geq \max\{k_1, k_2\} > K$, we see that the contribution from the second term is bounded by

$$\ll \frac{B \log B}{K} \sum_{a, b} \frac{\tau_3(|a^4 - b^4|)}{\sqrt{|a^2 - b^2|} \max\{a, b\}} \ll B,$$

by the proof of (4.2). This completes the proof of the lemma. \square

In view of this result we may now proceed under the assumption that $k_1, k_2 \leq K$ in (6.2). We will also run into trouble when estimating the contribution from various error terms when the 2-adic order of s in $L_{k_1, k_2}(B)$ is allowed to be very large. Let $\tilde{L}_{k_1, k_2}(B)$ be defined as for $L_{k_1, k_2}(B)$ but with the additional restriction that $\nu_2(s) \leq 2Z_2$. We easily adapt the proof of the preceding result to show that $L_{k_1, k_2}(B)$ may be replaced by $\tilde{L}_{k_1, k_2}(B)$ in (6.2), with an acceptable error.

We now apply Möbius inversion in (6.2), which thereby allows us to work with the equality

$$L(B) = \sum_{\substack{k_1 | (a^2 - b^2)/\lambda_1 \\ k_1 \leq K}}^b \sum_{\substack{k_2 | (a^2 + b^2)/\lambda_2 \\ k_2 \leq K}}^b \sum_{\ell \in \mathbb{N}}^b \mu(k_1) \mu(k_2) \mu(\ell) \tilde{L}_{k_1, k_2, \ell}(B). \quad (6.7)$$

Here K is given by (6.5) and $\tilde{L}_{k_1, k_2, \ell}(B)$ denotes the number of $(s, t) \in \Lambda(\mathbf{k}, \boldsymbol{\lambda}, \ell)$ such that (2), (3) and (6) hold in the definition of $L(B)$, with $2 \nmid \gcd(s, t)$ and $\nu_2(s) \leq 2Z_2$. Note that it will facilitate our ensuing investigation to restrict the summation to odd values of ℓ here.

When it comes to estimating $\tilde{L}_{k_1, k_2, \ell}(B)$ asymptotically as $B \rightarrow \infty$ we will encounter problems when ℓ is large. For given $T > 0$ and $Y \geq 1$, we now let $L_{k_1, k_2, \ell}^\dagger(Y, T)$ denote the number of

$$(s, t) \in \Lambda(\mathbf{k}, \boldsymbol{\lambda}, \ell) \cap \mathcal{R}^\dagger(2^\nu \lambda_1 \lambda_2 Y / \max\{a, b\})$$

for which $\gcd(s, t) > T$, where $\mathcal{R}^\dagger(X)$ is given by (5.8). Note that $\tilde{L}_{k_1, k_2, \ell}(B) \leq L_{k_1, k_2, \ell}^\dagger(B, 1/2)$ in the above notation. We have the following key result.

Lemma 15. — Suppose $T > 0$ and $A, Y \geq 1$, with $A \leq \sqrt{B}/Z_1^c$ and $Y \leq B^5$. Then we have

$$\sum_{\substack{(a, b) \in \mathbb{N}^2 \\ \gcd(a, b) = 1, ab \neq 1 \\ \nu_2(a^2 - b^2) \leq Z_2 \\ \max\{a, b\} \leq A}} \sum_{\nu \leq Z_2} \sum_{\substack{k_1 \lambda_1 | a^2 - b^2 \\ k_1 \leq K}}^b \sum_{\substack{k_2 \lambda_2 | a^2 + b^2 \\ k_2 \leq K}}^b \sum_{\ell \in \mathbb{N}}^b L_{k_1, k_2, \ell}^\dagger(Y; T) \ll A^2 Z_1^{O(1)}[T] + \frac{Y Z_1^{O(1)}}{T},$$

where $[T]$ denotes the integer part of T .

Proof. — The idea is to reintroduce a coprimality condition on the (s, t) that are to be counted. Since we are only interested in an upper bound we may drop any of the defining conditions in $L_{k_1, k_2, \ell}^\dagger(Y; T)$ that we care to choose. In this way it is clear that

$$L_{k_1, k_2, \ell}^\dagger(Y; T) \leq \sum_{L > T} \# \left\{ (s, t) \in \Lambda(\mathbf{k}, \boldsymbol{\lambda}, \ell) \cap \mathcal{R}^\dagger\left(\frac{2^\nu \lambda_1 \lambda_2 Y}{\max\{a, b\}}\right) : \gcd(s, t) = L \right\}.$$

Let

$$\lambda'_1 := k_1 \lambda_1, \quad \lambda'_2 := k_2 \lambda_2 \tag{6.8}$$

and

$$\lambda''_1 := \frac{\lambda'_1}{\gcd(\lambda'_1, L)}, \quad \lambda''_2 := \frac{\lambda'_2}{\gcd(\lambda'_2, L)}. \tag{6.9}$$

Note that the summand is zero unless $\ell \mid L$. Making the change of variables $(s, t) = L(s', t')$ with $\gcd(s', t') = 1$, we conclude that the summand is bounded by the number of coprime vectors $(s', t') \in \mathbb{Z}^2$ such that

$$\lambda''_1 \mid s', \quad \lambda''_2 \mid s' - t'a, \tag{6.10}$$

with $(s', t') \in \mathcal{R}^\dagger(2^\nu \lambda_1 \lambda_2 Y / (L^2 \max\{a, b\}))$.

Recall the definition (5.1) of Q_1 and Q_2 and let $\lambda^+ := \gcd(Q_1(s', t'), Q_2(s', t'))$. Since $\gcd(s', t') = 1$ it follows that

$$\lambda^+ = \gcd(4s'(s' - t'a), t'^2(a^2 - b^2) - 2s'^2) \geq \gcd(s', a^2 - b^2) \gcd(s' - t'a, a^2 + b^2).$$

In view of the fact that $\gcd(\lambda_1'', \lambda_2'') = 1$ and (6.10) holds, we obtain $\lambda^+ \geq \lambda_1 \lambda_2 / M$, with $M := \gcd(\lambda_1 \lambda_2, L)$. In particular $M \mid a^4 - b^4$. It now follows that there exists an absolute constant $c > 0$ such that

$$0 < -\frac{\max\{a, b\} Q_j(s', t')}{\lambda^+} \leq c \frac{2^{Z_2-1} M Y}{L^2} \leq c \frac{M Y \log B}{L^2},$$

for $j = 1, 2$. Writing $L = M L'$, we deduce that

$$(s', t') \in \mathcal{R}^\dagger \left(\frac{\lambda^+ Y'}{\max\{a, b\}} \right), \quad Y' := c \frac{Y \log B}{M L'^2}. \quad (6.11)$$

We note here that $Y' \gg \max\{a, b\}$.

Let us write $E(Y)$ for the term that is to be estimated in the statement of the lemma. We will employ the trivial estimate $\tau(n) \ll Z_1^{O(1)}$, for any $n \leq B^c$. Hence in $E(Y)$ there are clearly at most $\tau_3(|a^2 - b^2|) \tau_3(a^2 + b^2) \tau(|a^4 - b^4|) \ll Z_1^{O(1)}$ possible values of $k_1, k_2, \lambda_1, \lambda_2$ and M . Furthermore there are also at most $\tau(L) \ll Z_1^{O(1)}$ possible values of ℓ such that $\ell \mid L$. On noting that the summation over ν contributes $O(Z_1)$, we conclude from the proof of Lemma 10 that

$$\begin{aligned} E(Y) &\ll Z_1^{O(1)} \sum_{\substack{a, b < A \\ \gcd(a, b) = 1 \\ ab \neq 1}} \max_{M \ll A^4} \sum_{L' > T/M} \# \left\{ (s', t') \in \mathbb{Z}^2 : \begin{array}{l} \gcd(s', t') = 1, \\ (6.11) \text{ holds} \end{array} \right\} \\ &\ll Z_1^{O(1)} \sum_{\substack{a, b < A \\ \gcd(a, b) = 1 \\ ab \neq 1}} \max_{M \ll A^4} \sum_{L' > T/M} \widehat{M}_{a, b}(Y'), \end{aligned} \quad (6.12)$$

where $\widehat{M}_{a, b}(Y')$ is given by (3.3). Here we have used $s'(s' - t'a) \neq 0$ appearing in $\mathcal{R}^\dagger(X)$ to ensure $xy \neq 1$ in $\widehat{M}_{a, b}(Y')$. Moreover it is useful to note that $\widehat{M}_{a, b}(Y') \geq 1$ for $Y' \gg \max\{a, b\}$. Let us write $E_1(Y)$, respectively $E_2(Y)$, for the overall contribution to the right hand side from a, b, L' such that $\max\{a, b\} \leq \sqrt{Y'}$, respectively $\max\{a, b\} > \sqrt{Y'}$. To begin with it follows from Lemma 8 that

$$E_1(Y) \ll Z_1^{O(1)} \max_M \sum_{L' > T/M} Y' (\log B)^4 \ll \frac{Y Z_1^{O(1)}}{T}.$$

It remains to estimate $E_2(Y)$. We return to (6.12), now with $\sqrt{Y'} < \max\{a, b\} < A$ and $ab \neq 1$ in the summation over a, b . We will write $E_{2,1}(Y)$ for the contribution to the right hand side from values of $L' \leq T$, and $E_{2,2}(Y)$ for the contribution from values of $L' > T$. Beginning with the former, we note that $E_{2,1}(Y) = 0$ if $T < 1$. If on the other hand $T \geq 1$ then an application of (4.1) gives

$$\begin{aligned} E_{2,1}(Y) &\ll Z_1^{O(1)} \max_M \sum_{T/M < L' \leq T} \sum_{a, b < A} \left(1 + \frac{Y'}{\max\{a, b\}^{5/3} |a - b|^{1/3}} \right) \\ &\ll A^2 Z_1^{O(1)} T + \frac{Y Z_1^{O(1)}}{T} \end{aligned}$$

This is satisfactory for the lemma.

Finally we must deal with $E_{2,2}(Y)$. For this we will reverse the roles of the variables a, b and x, y in $\widehat{M}_{a,b}(Y')$. We have

$$E_{2,2}(Y) \ll Z_1^{O(1)} \max_M \sum_{L' > T} \# \left\{ a, b, x, y, z : \begin{array}{l} \gcd(a, b) = \gcd(x, y) = 1, \\ ab, xy \neq 1, \\ \sqrt{Y'} < \max\{a, b\} < A, \\ (a^2 - b^2)x^2 + (a^2 + b^2)y^2 = 2z^2, \\ \max\{a, b\} \max\{x, y\} \leq Y' \end{array} \right\},$$

On writing $Y'' := cY(\log B)/L'^2$, it now follows from Lemma 8 that

$$E_{2,2}(Y) \ll Z_1^{O(1)} \max_M \sum_{L' > T} \sum_{\substack{x, y \leq \sqrt{Y''} \\ \gcd(x, y) = 1 \\ xy \neq 1}} M_{y,x}(Y'') \ll Y Z_1^{O(1)} \sum_{L' > T} \frac{1}{L'^2} \ll \frac{Y Z_1^{O(1)}}{T}.$$

This is satisfactory and therefore completes the proof of the lemma. \square

We would now like to eliminate the contribution to (6.7) from large ℓ . Taking $Y = B, A = Z_1^{-c}\sqrt{B}$ and $T = Z_1^{c/10}$ in Lemma 15 we deduce that the overall contribution from $\ell > Z_1^{c/10}$ is

$$\ll \frac{B Z_1^{O(1)}}{Z_1^{c/10}}.$$

This estimate exhibits a feature common to much of what follows. The notation $Z_1^{O(1)}$ means that we allow an arbitrary, but absolutely bounded, power of Z_1 in the error term. The term $Z_1^{c/10}$ should be thought of as a parameter since we are free to take c large enough to nullify the effect of the numerator. Drawing this observation together with (5.9), our argument so far has established the following result.

Lemma 16. — *We have*

$$\begin{aligned} N_1(B) &= 2 \sum_{(a,b) \in \mathcal{A}_1} \sum_{\nu \leq \max\{1, \nu_2(a^2 - b^2)\}} \sum_{\substack{k_1 \lambda_1 | a^2 - b^2 \\ k_1 \leq K}}^{\flat} \sum_{\substack{k_2 \lambda_2 | a^2 + b^2 \\ k_2 \leq K}}^{\flat} \\ &\quad \times \sum_{\substack{\ell \leq Z_1^{c/10} \\ \ell \leq Z_1^{c/10}}}^{\flat} \mu(k_1) \mu(k_2) \mu(\ell) \tilde{L}_{k_1, k_2, \ell}(B) + O\left(\frac{B(\log B)^4}{\log \log B}\right), \end{aligned}$$

where \mathcal{A}_1 is given by (4.4) and K by (6.5).

7. Lattice point counting

Our task in this section is to set the scene for an asymptotic formula for the quantity $\tilde{L}_{k_1, k_2, \ell}(B)$. Let $\Lambda = \Lambda(\mathbf{k}, \boldsymbol{\lambda}, \ell) \subset \mathbb{Z}^2$ be the lattice defined in (6.1). On writing

$$\mathcal{R}'_\nu := \mathcal{R}'\left(\frac{2^\nu \lambda_1 \lambda_2 B}{\max\{a, b\}}\right), \quad (7.1)$$

where $\mathcal{R}'(X)$ is defined by (5.7), it follows from the previous section that

$$\tilde{L}_{k_1, k_2, \ell}(B) = \# \{(s, t) \in \Lambda \cap \mathcal{R}'_\nu : (5.3) \text{ holds, } 2 \nmid \gcd(s, t), \nu_2(s) \leq 2Z_2\}. \quad (7.2)$$

We now come to a rather delicate feature of the proof. Ignoring the supplementary 2-adic conditions in our new expression for $\tilde{L}_{k_1, k_2, \ell}(B)$, the obvious next step would be to try and approximate this cardinality with something like the volume of the region in question, divided by the determinant of Λ . Ignoring also the contribution from the error term in this approximation, one would then be led to sum this quantity over all of the remaining parameters. An absolutely crucial observation here is the following: for some ranges of the parameters λ_1, λ_2 we will have $\tilde{L}_{k_1, k_2, \ell}(B) = 0$ in Lemma 16, even if the approximation $\text{vol}(\mathcal{R}'_\nu)/\det \Lambda$ is non-zero. Thus a further reduction on the set of allowable parameters λ_1, λ_2 is necessary.

For any $R > 0$ and $(a, b) \in \mathcal{A}_1$, define the region

$$V_{a, b}(R) := \left\{ (t_1, t_2) \in \mathbb{R}_{\geq 1}^2 : \begin{array}{l} \max\{a, b\}t_1 \leq Rt_2, \\ \max\{a, b\}t_2 \leq Rt_1, \\ \max\{a, b\}^3/R \leq t_1t_2, \\ t_1t_2 \leq \max\{a, b\}R \end{array} \right\}. \quad (7.3)$$

We will show how the summation over λ_1, λ_2 is necessarily restricted to this set for a suitable choice of R .

It follows from the definition of \mathcal{A}_1 that $\min\{a, b, |a - b|\} \geq \max\{a, b\}/Z_2^2$. Taken together with Lemma 13, the fact that we are interested in integers $(s, t) \in \Lambda \cap \mathcal{R}'_\nu$ therefore implies that

$$1 \leq |s| \leq c\sqrt{X}, \quad 1 \leq t \leq c \frac{\sqrt{X}Z_2}{\max\{a, b\}}, \quad 1 \leq |s - at| \leq c\sqrt{X}Z_2,$$

with X given by (6.6). The middle inequality here implies that

$$\max\{a, b\}^3 \leq c2^\nu BZ_2^2 \lambda_1 \lambda_2,$$

in which we recall our convention that c is used to denote a generic absolute positive constant. But we also know that $\lambda_1 \mid s$ and $\lambda_2 \mid s - at$, so that we may take $|s| \geq \lambda_1$ and $|s - at| \geq \lambda_2$ in the first and third inequality, giving

$$\max\{a, b\} \lambda_1 \leq c2^\nu B \lambda_2, \quad \max\{a, b\} \lambda_2 \leq c2^\nu BZ_2^2 \lambda_1.$$

The final inequality that we seek to establish is

$$\lambda_1 \lambda_2 \leq c \max\{a, b\} 2^\nu BZ_2^6. \quad (7.4)$$

This is rather more subtle and arises from an inherent symmetry between λ_1, λ_2 and μ_1, μ_2 , where the latter are non-zero integers such that

$$\lambda_1 \mu_1 = a^2 - b^2, \quad \lambda_2 \mu_2 = a^2 + b^2. \quad (7.5)$$

In particular all of $\lambda_1, \lambda_2, \mu_1, \mu_2$ are coprime to a and b . It follows from the divisibility information on s and $s - at$ that there exist non-zero integers σ, τ such that $s = \lambda_1 \sigma$

and $s - at = \lambda_2 \tau$. The height conditions for $s, s - at$ imply that

$$\begin{aligned} 1 \leq \sigma^2 &\leq c \frac{2^\nu B \lambda_2}{\max\{a, b\} \lambda_1} = c \frac{2^\nu B(a^2 + b^2) |\mu_1|}{\max\{a, b\} |a^2 - b^2| \mu_2} \leq c \frac{2^\nu B Z_2^2 |\mu_1|}{\max\{a, b\} \mu_2}, \\ 1 \leq \tau^2 &\leq c \frac{2^\nu B Z_2^2 \lambda_1}{\max\{a, b\} \lambda_2} = c \frac{2^\nu B Z_2^2 |a^2 - b^2| \mu_2}{\max\{a, b\} (a^2 + b^2) |\mu_1|} \leq c \frac{2^\nu B Z_2^2 \mu_2}{\max\{a, b\} |\mu_1|}. \end{aligned}$$

Furthermore, we have

$$\mu_1 \mu_2 a t = \mu_1 \mu_2 (\lambda_1 \sigma - \lambda_2 \tau) = \mu_2 (a^2 - b^2) \sigma - \mu_1 (a^2 + b^2) \tau. \quad (7.6)$$

In particular

$$\mu_2 \sigma + \mu_1 \tau \equiv 0 \pmod{a}, \quad (7.7)$$

since a, b are coprime. Now we have $\mu_2 \sigma + \mu_1 \tau = 0$ if and only if $\sigma = \mu_1 x$ and $\tau = -\mu_2 x$ for some non-zero integer x . Using (7.6) we easily deduce that

$$\frac{s}{t} = \frac{a^2 - b^2}{2a},$$

which contradicts the hypotheses in the definition of (5.7). Hence $\mu_2 \sigma + \mu_1 \tau \neq 0$ in (7.7), giving the further inequality

$$\frac{\max\{a, b\}}{Z_2^2} \leq a \leq |\mu_2 \sigma| + |\mu_1 \tau| \leq c \sqrt{\frac{2^\nu B Z_2^2 |\mu_1| \mu_2}{\max\{a, b\}}},$$

whence

$$\frac{\max\{a, b\}^3}{Z_2^6} \leq c 2^\nu B |\mu_1| \mu_2 = c 2^\nu B \frac{|a^4 - b^4|}{\lambda_1 \lambda_2} \leq c 2^\nu B \frac{\max\{a, b\}^4}{\lambda_1 \lambda_2}.$$

This therefore implies (7.4).

Bringing our argument together we have therefore shown that the summation over λ_1, λ_2 in Lemma 16 is subject to $(\lambda_1, \lambda_2) \in V_{a,b}(c 2^\nu B Z_2^6)$. In fact, in view of the bounds $k_1, k_2 \leq K$, it is trivial to see that

$$(k_1 \lambda_1, k_2 \lambda_2) \in V_{a,b}(c B K^3).$$

Here, using the definitions of K and Z_2 , we have been able to replace $2^\nu Z_2^6$ by an additional factor of K . The following result refines this somewhat.

Lemma 17. — *We have*

$$\begin{aligned} N_1(B) &= 2 \sum_{(a,b) \in \mathcal{A}_1} \sum_{\nu \leq \max\{1, \nu_2(a^2 - b^2)\}} \sum_{\substack{k_1 \lambda_1 | a^2 - b^2 \\ k_1 \leq K}} \sum_{\substack{k_2 \lambda_2 | a^2 + b^2 \\ k_2 \leq K}} \chi_{k_1 \lambda_1, k_2 \lambda_2}(B/K) \\ &\quad \times \sum_{\substack{\ell \leq Z_1^{c/10} \\ \ell \leq Z_1^{c/10}}} \mu(k_1) \mu(k_2) \mu(\ell) \tilde{L}_{k_1, k_2, \ell}(B) + O\left(\frac{B(\log B)^4}{\log \log B}\right), \end{aligned}$$

where \mathcal{A}_1 is given by (4.4), K by (6.5) and $\chi_{t_1, t_2}(R)$ is the characteristic function of the region (7.3).

Proof. — We have already shown that the lemma is true with $\chi_{k_1\lambda_1, k_2\lambda_2}(cBK^3)$ in place of $\chi_{k_1\lambda_1, k_2\lambda_2}(B/K)$. To establish the lemma it suffices to estimate the overall contribution to the main term from values of $k_1, k_2, \lambda_1, \lambda_2$ for which

$$(k_1\lambda_1, k_2\lambda_2) \in V_{a,b}(cBK^3) \setminus V_{a,b}(B/K),$$

in the notation of (7.3). This forces λ_1, λ_2 to satisfy one of four further inequalities. Let us show how to handle the contribution corresponding to λ_1, λ_2 satisfying

$$\frac{Bk_1\lambda_1}{k_2K \max\{a, b\}} < \lambda_2 \leq \frac{cBK^3k_1\lambda_1}{k_2 \max\{a, b\}}, \quad (7.8)$$

the remaining cases being handled in an identical manner.

Let $L_1(B)$ denote the contribution to $\tilde{L}_{k_1, k_2, \ell}(B)$ arising from $(s, t) \in \mathbb{Z}^2$ for which $\gcd(s, t) \leq Z_1^{c/10}$. Lemma 15 implies that the overall contribution to Lemma 17 from the remaining quantity is $O(BZ_1^{O(1)}/Z_1^{c/10})$, which is satisfactory. To estimate $L_1(B)$, we write $(s, t) = L(s', t')$ with $\gcd(s', t') = 1$ and $L \leq Z_1^{c/10}$ an odd integer divisible by ℓ . The summation over ν allows us to assume that $2^\nu \mid 4s'$. Let us write λ'_i, λ''_i as in (6.8) and (6.9). We conclude from Lemmas 1 and 13 that

$$L_1(B) \ll \sum_{\substack{L \leq Z_1^{c/10} \\ \ell \mid L}}^b \left(1 + \frac{2^\nu \lambda_1 \lambda_2 B Z_2}{L^2 (\det \Lambda') \max\{a, b\}^2} \right),$$

where now $\Lambda' \subseteq \mathbb{Z}^2$ is the lattice of $(s', t') \in \mathbb{Z}^2$ for which

$$\lambda''_1 \mid s', \quad \lambda''_2 \mid s' - at', \quad 2^\nu \mid 4s'.$$

Lemma 12 therefore leads us to an overall contribution of

$$\ll \sum_{(a,b) \in \mathcal{A}_1} \sum_{\nu \leq Z_2} \sum_{k_i, \lambda_i}^b \sum_{L \leq Z_1^{c/10}}^b \sum_{\ell \mid L} \left(1 + \frac{\gcd(k_1 k_2 \lambda_1 \lambda_2, L) B Z_2}{L^2 k_1 k_2 \max\{a, b\}^2} \right).$$

Here the summation over k_i, λ_i is subject to the restriction that they should be odd, with $k_i \lambda_i \mid a^2 + (-1)^i b^2$, $k_i \leq K$ and (7.8) holding.

For fixed a, b and L there are at most $\tau_3(|a^4 - b^4|) \tau(L) Z_2 = O(Z_1^{O(1)})$ values of $\nu, k_i, \lambda_i, \ell$. Hence the overall contribution from the first term in the above summand is $O(BZ_1^{O(1)}/Z_1^c)$, which is satisfactory. The remaining contribution is clearly

$$\ll B(\log \log B)^2 \sum_{(a,b) \in \mathcal{A}_1} \sum_{k_i, \lambda_i}^b \sum_{L \leq Z_1^{c/10}}^b \sum_{\ell \mid L} \frac{\gcd(k_1 k_2 \lambda_1 \lambda_2, L)}{L^2 k_1 k_2 \max\{a, b\}^2},$$

on carrying out the summation over ν . We now break the summation over L into $L \leq K$ and $K < L \leq Z_1^{c/10}$, with K given by (6.5). We see that the overall contribution

from $L > K$ is

$$\begin{aligned} &\ll \frac{B(\log B)(\log \log B)^2}{K} \sum_{(a,b) \in \mathcal{A}_1} \sum_{k_i, \lambda_i}^b \frac{\tau(k_1 k_2 \lambda_1 \lambda_2)}{k_1 k_2 \max\{a, b\}^2} \\ &\ll \frac{B(\log B)(\log \log B)^2}{K} \sum_{(a,b) \in \mathcal{A}_1} \frac{\tau(|a^4 - b^4|) \tau_3(|a^4 - b^4|)}{\max\{a, b\}^2} \\ &\ll B, \end{aligned}$$

which is satisfactory. Turning to $L \leq K$, we take $\gcd(k_1 k_2 \lambda_1 \lambda_2, L) \leq L$ and so obtain the overall contribution

$$\ll B(\log \log B)^4 \sum_{(a,b) \in \mathcal{A}_1} \sum_{k_i, \lambda_i}^b \frac{1}{k_1 k_2 \max\{a, b\}^2}. \quad (7.9)$$

We will find it convenient to proceed under the additional hypothesis

$$\lambda_1 \lambda_2 \leq \sqrt{2} \max\{a, b\}^2. \quad (7.10)$$

Underlying this assumption is the symmetry that exists between λ_i and μ_i such that (7.5) holds. Thus if (7.10) fails then it is immediately clear from the definition (4.4) of \mathcal{A}_1 that

$$|\mu_1| \mu_2 \leq \sqrt{2} \max\{a, b\}^2.$$

Furthermore, (7.8) translates into

$$\frac{\max\{a, b\}(a^2 + b^2)k_2|\mu_1|}{cBk_1K^3|a^2 - b^2|} \leq \mu_2 < \frac{\max\{a, b\}(a^2 + b^2)k_2K|\mu_1|}{Bk_1|a^2 - b^2|}$$

which also shares the same structure. This discussion shows that we may proceed under the assumption that (7.10) holds in what follows.

We now break the summation over a, b into dyadic intervals, in order to fix attention on the range $A \leq \max\{a, b\} < 2A$ for $A \leq \sqrt{B}/Z_1^c$. Interchanging the order of summation (7.9) becomes

$$\ll B(\log \log B)^4 \sum_A \frac{1}{A^2} \sum_{k_i, \lambda_i}^b \frac{1}{k_1 k_2} N(A),$$

where

$$N(A) := \# \{ (a, b) \in \mathcal{A}_1 : A \leq \max\{a, b\} < 2A, \ k_i \lambda_i \mid a^2 + (-1)^i b^2 \}.$$

Furthermore, the summation over $k_i, \lambda_i \in \mathbb{N}$ is over odd integers subject to $k_i \leq K$ and

$$\frac{B\lambda_1}{2AK^2} \leq \frac{Bk_1\lambda_1}{2k_2AK} < \lambda_2 \leq \frac{cBK^3k_1\lambda_1}{k_2A} \leq \frac{cBK^4\lambda_1}{A}. \quad (7.11)$$

Let $\varrho_i(q)$ be the number of square roots of $(-1)^{i+1}$ modulo q . For given q_1, q_2 the conditions $q_1 \mid a^2 - b^2$ and $q_2 \mid a^2 + b^2$ force the vector (a, b) to lie on one of at most $\varrho_1(q_1)\varrho_2(q_2)$ integer sublattices of \mathbb{Z}^2 , each of determinant $q_1 q_2$. Now it is easy to see that $\varrho_1(q_1) \leq 2^{\omega(q_1)}$ and

$$\varrho_2(q_2) \leq \sum_{e \mid q_2} |\mu(e)| \chi(e) \leq r(q_2),$$

where χ is the real non-principal character modulo 4 and r denotes the sums of two squares function. It follows that $\varrho_2(rq) \leq 2^{\omega(r)} r(q)$ for any $r, q \in \mathbb{N}$. Since a, b lie in a region in \mathbb{R}^2 of volume $O(A^2)$, an application of Lemma 1 furnishes the contribution

$$\begin{aligned} &\ll B(\log \log B)^4 \sum_A \frac{1}{A^2} \sum_{k_i, \lambda_i}^b \frac{2^{\omega(k_1 k_2)}}{k_1 k_2} 2^{\omega(\lambda_1)} r(\lambda_2) \left(\frac{A^2}{k_1 k_2 \lambda_1 \lambda_2} + 1 \right) \\ &\ll B(\log \log B)^4 (\log K)^4 \sum_A \frac{1}{A^2} \sum_{\lambda_1, \lambda_2}^b 2^{\omega(\lambda_1)} r(\lambda_2) \left(\frac{A^2}{\lambda_1 \lambda_2} + 1 \right), \end{aligned}$$

on summing over $k_1, k_2 \leq K$. Here the inner summation is over λ_1, λ_2 such that

$$\frac{B\lambda_1}{2AK^2} < \lambda_2 \leq \frac{cBK^4\lambda_1}{A}, \quad \lambda_1 \lambda_2 \ll A^2,$$

as follows from (7.10) and (7.11). Thus we easily obtain the overall contribution

$$\begin{aligned} &\ll B(\log \log B)^8 \sum_A \sum_{\lambda_1, \lambda_2}^b \frac{2^{\omega(\lambda_1)} r(\lambda_2)}{\lambda_1 \lambda_2} \ll B(\log \log B)^8 \sum_A \sum_{\lambda_1}^b \frac{2^{\omega(\lambda_1)}}{\lambda_1} \log(2cK^6) \\ &\ll B(\log B)^3 (\log \log B)^9, \end{aligned}$$

on summing first over λ_2 , then over λ_1 and finally over the $O(\log B)$ choices for A . This completes the proof of the lemma. \square

Let $a, b, \nu, k_i, \lambda_i, \ell$ be an arbitrary choice of parameters that appear in the main term in Lemma 17's estimate for $N_1(B)$. We now come to our estimation of (7.2), where $\Lambda = \Lambda(\mathbf{k}, \boldsymbol{\lambda}, \ell) \subset \mathbb{Z}^2$ is the lattice defined in (6.1) and $\mathcal{R}'_\nu \subset \mathbb{R}^2$ is given by (7.1). Given that \mathcal{R}'_ν is clearly defined with piecewise continuous boundary, we would like to apply the well-known formula (1.8). However, there are two complications that prevent a routine application of this estimate. Firstly, we will need to take care of the 2-adic conditions apparent in (7.2). Secondly we need to deal with the fact that once summed over the remaining parameters, the error term in the above asymptotic formula for $\#(\Lambda \cap \mathcal{R}'_\nu)$ does not make a satisfactory overall contribution from the point of view of the main theorem.

In spite of these objections we will dedicate the remainder of this section to interpreting the main term in (1.8) in the present context, delaying our discussion of the error term until the subsequent section. Let us consider the first issue mentioned above, namely the 2-adic conditions on s, t . We retain the shorthand notation $\Lambda = \Lambda(\mathbf{k}, \boldsymbol{\lambda}, \ell)$ and make the observation that $\text{vol}(\mathcal{R}'_\nu) = 2^\nu \text{vol}(\mathcal{R}'_0)$, for any $\nu \geq 0$. Furthermore, we will set

$$\Lambda_{i,j} := \{(s, t) \in \Lambda : 2^i \mid s, 2^j \mid t\},$$

for $i, j \geq 0$. Since the parameters in Λ are all odd it easily follows from Lemma 12 that $\det \Lambda_{i,j} = 2^{i+j} \det \Lambda$. It is here that our earlier restriction to odd values of ℓ pays dividends. Finally, it will be convenient to define

$$\pi_k := \begin{cases} 1, & \text{if } 2 \mid k, \\ 0, & \text{if } 2 \nmid k, \end{cases} \quad (7.12)$$

for any $k \in \mathbb{N}$.

We will separate our investigation according to the value of ν . We suppress the dependence on k_1, k_2, ℓ in the expression for $\tilde{L}_{k_1, k_2, \ell}(B)$ in (7.2), replacing it by $L^\nu(B)$ in order to underline the dependence on ν . We have 3 basic possibilities to consider: either $\nu = 0$ or $\nu = 1$ or $\nu \geq 3$. Note that (5.3) ensures that the possibility $\nu = 2$ does not arise.

Let us begin by supposing that $\nu = 0$, which according to (5.3) is only possible when $2 \mid ab$ and $2 \nmid t$, so that $\pi_{ab} = 1$. Recall the notation introduced above for $\Lambda_{i,j}$. It follows that

$$\begin{aligned} L^0(B) &= \sum_{0 \leq \sigma \leq 2Z_2} \# \{(s, t) \in \Lambda \cap \mathcal{R}'_0 : 2^\sigma \parallel s, 2 \nmid t\} \\ &= \sum_{0 \leq \sigma \leq 2Z_2} \sum_{i \geq 0} \sum_{j \geq 0} \mu(2^i) \mu(2^j) \#(\Lambda_{\sigma+i,j} \cap \mathcal{R}'_0). \end{aligned}$$

In line with (1.8) we expect the cardinality in the summand to satisfy an asymptotic formula with main term

$$\frac{\text{vol}(\mathcal{R}'_0)}{\det \Lambda_{\sigma+i,j}} = \frac{\text{vol}(\mathcal{R}'_0)}{2^{\sigma+i+j} \det \Lambda}.$$

We may now conclude as follows.

Lemma 18. — *We have*

$$L^0(B) = \pi_{ab} \left(\frac{1}{2} - \frac{1}{2^{[2Z_2]}} \right) \frac{\text{vol}(\mathcal{R}'_0)}{\det \Lambda} + E^0(B),$$

where

$$E^0(B) := \pi_{ab} \sum_{0 \leq \sigma \leq 2Z_2} \sum_{i,j \geq 0} \mu(2^i) \mu(2^j) \left(\#(\Lambda_{\sigma+i,j} \cap \mathcal{R}'_0) - \frac{\text{vol}(\mathcal{R}'_0)}{\det \Lambda_{\sigma+i,j}} \right).$$

We have found it useful to include π_{ab} in the main term for this estimate, to help keep track of the fact that we are only interested in the value of $L^0(B)$ when $2 \mid ab$. When $\nu = 1$ it follows from (5.3) that $2 \mid t$ if $2 \mid ab$ and $2 \nmid s$ if $2 \nmid ab$. Hence (7.2) yields

$$L^1(B) = \pi_{ab} \# \{(s, t) \in \Lambda \cap \mathcal{R}'_1 : 2 \nmid s, 2 \mid t\} + \pi_{ab+1} \# \{(s, t) \in \Lambda \cap \mathcal{R}'_1 : 2 \mid s\}.$$

Arguing as above, we now have

$$L^1(B) = \pi_{ab} \sum_{i \geq 0} \mu(2^i) \#(\Lambda_{i,1} \cap \mathcal{R}'_1) + \pi_{ab+1} \sum_{i \geq 0} \mu(2^i) \#(\Lambda_{i,0} \cap \mathcal{R}'_1).$$

Drawing out the obvious main term, as previously, we therefore conclude the proof of the following result.

Lemma 19. — *We have*

$$L^1(B) = \left(1 - \frac{\pi_{ab}}{2} \right) \frac{\text{vol}(\mathcal{R}'_0)}{\det \Lambda} + E_\alpha^1(B) + E_\beta^1(B),$$

where

$$E_\alpha^1(B) := \pi_{ab} \sum_{i \geq 0} \mu(2^i) \left(\#(\Lambda_{i,1} \cap \mathcal{R}'_1) - \frac{\text{vol}(\mathcal{R}'_1)}{\det \Lambda_{i,1}} \right),$$

$$E_\beta^1(B) := \pi_{ab+1} \sum_{i \geq 0} \mu(2^i) \left(\#(\Lambda_{i,0} \cap \mathcal{R}'_1) - \frac{\text{vol}(\mathcal{R}'_1)}{\det \Lambda_{i,0}} \right).$$

Assume now that $\nu \geq 3$, under which assumption (5.3) implies that $2 \nmid ab$ and $2 \mid s$, with $\nu = \min\{2 + \nu_2(s), \nu_2(a^2 - b^2)\}$. In particular, if $\nu_2(s) \leq \nu_2(a^2 - b^2) - 2$ then it follows that $\nu_2(s) = \nu - 2$. This is then automatically bounded above by $2Z_2$. Otherwise we must have $\nu_2(s) \geq \nu_2(a^2 - b^2) - 1$ and $\nu_2(a^2 - b^2) = \nu$. Moreover, we must force $2 \nmid t$ in our considerations. We deduce from (7.2) that $L^\nu(B)$ is the sum of

$$T_1 := \pi_{ab+1} \sum_{i,j \geq 0} \mu(2^i) \mu(2^j) \#(\Lambda_{\nu-2+i,j} \cap \mathcal{R}'_\nu)$$

and

$$T_2 := \pi_{ab+1} \sum_{\nu-1 \leq \sigma \leq 2Z_2} \sum_{i,j \geq 0} \mu(2^i) \mu(2^j) \#(\Lambda_{\sigma+i,j} \cap \mathcal{R}'_\nu).$$

Note the first term is always present, since $\nu \leq \max\{1, \nu_2(a^2 - b^2)\}$, but the latter term only appears if $\nu = \nu_2(a^2 - b^2)$. The following result is now available.

Lemma 20. — *Let $\nu \geq 3$. Then we have*

$$L^\nu(B) = \pi_{ab+1} \frac{\text{vol}(\mathcal{R}'_0)}{\det \Lambda} + E_\alpha^\nu(B),$$

if $\nu < \nu_2(a^2 - b^2)$ and

$$L^\nu(B) = \pi_{ab+1} \left(2 - \frac{2^\nu}{2^{[2Z_2]+2}} \right) \frac{\text{vol}(\mathcal{R}'_0)}{\det \Lambda} + E_\alpha^\nu(B) + E_\beta^\nu(B),$$

if $\nu = \nu_2(a^2 - b^2)$, where

$$E_\alpha^\nu(B) := \pi_{ab+1} \sum_{i,j \geq 0} \mu(2^i) \mu(2^j) \left(\#(\Lambda_{\nu-2+i,j} \cap \mathcal{R}'_\nu) - \frac{\text{vol}(\mathcal{R}'_\nu)}{\det \Lambda_{\nu-2+i,j}} \right),$$

$$E_\beta^\nu(B) := \pi_{ab+1} \sum_{\nu-1 \leq \sigma \leq 2Z_2} \sum_{i,j \geq 0} \mu(2^i) \mu(2^j) \left(\#(\Lambda_{\sigma+i,j} \cap \mathcal{R}'_\nu) - \frac{\text{vol}(\mathcal{R}'_\nu)}{\det \Lambda_{\sigma+i,j}} \right).$$

8. The error terms

It is now time to establish that the error terms in Lemmas 18, 19 and 20 make a satisfactory overall contribution once summed up over the relevant parameters appearing in Lemma 17. In order not to be encumbered with superfluous notation let us fix our attention on the error term

$$E(B) := \#(\Lambda \cap \mathcal{R}'(X)) - \frac{\text{vol}(\mathcal{R}'(X))}{\det \Lambda},$$

where X is given by (6.6). The outcome of our investigation will be a proof of the following result.

Lemma 21. — *Let*

$$\mathcal{E}(B) := \sum_{(a,b) \in \mathcal{A}_1} \sum_{\nu} \sum_{\substack{k_1 \lambda_1 | a^2 - b^2 \\ k_1 \leq K}}^b \sum_{\substack{k_2 \lambda_2 | a^2 + b^2 \\ k_2 \leq K}}^b \sum_{\ell \leq Z_1^{c/10}}^b \chi_{k_1 \lambda_1, k_2 \lambda_2}(B/K) |E(B)|,$$

where K is given by (6.5) and the ν summation is over $\nu \leq \max\{1, \nu_2(a^2 - b^2)\}$. Then we have

$$\mathcal{E}(B) \ll B \frac{Z_1^{O(1)}}{Z_1^c} + B(\log B)^3 (\log \log B)^2.$$

There is a degree of over simplification here. Indeed, for $\nu \geq 0$ we are actually interested in controlling the overall contribution from the error terms $E^0(B)$, $E_\alpha^\nu(B)$ and $E_\beta^\nu(B)$ that appear in Lemmas 18–20. However, one readily checks that the argument goes through for each of these. The crucial fact is that the summation over i, j, σ in the true error terms is over $i, j \leq 1$ and $\sigma \leq 2Z_2$. Hence at the expense of a harmless extra factor of $\log \log B$ in Lemma 21 one finds that the overall contribution from is satisfactory.

The proof of Lemma 21 is long and technical. We begin with some general facts about approximating the characteristic function of suitable bounded regions $\mathcal{S} \subset \mathbb{R}^2$ by smooth Gaussian weights. For any $\mathbf{x} \in \mathbb{R}^2$ let

$$B(\mathbf{x}, r) := \{\mathbf{y} \in \mathbb{R}^2 : \|\mathbf{y} - \mathbf{x}\| \leq r\}$$

denote the ball centered on \mathbf{x} with radius r , where $\|\mathbf{z}\| = \sqrt{z_1^2 + z_2^2}$ denotes the Euclidean norm of a vector $\mathbf{z} \in \mathbb{R}^2$. Let $H \geq 1$ be a parameter at our disposal. Define

$$S_- := \mathbb{R}^2 \setminus \bigcup_{\mathbf{x} \notin \mathcal{S}} B\left(\mathbf{x}, \frac{1}{\sqrt{H}}\right), \quad S_+ := \bigcup_{\mathbf{x} \in \mathcal{S}} B\left(\mathbf{x}, \frac{1}{\sqrt{H}}\right),$$

and notice that $S_- \subseteq \mathcal{S} \subseteq S_+$. We introduce the infinitely differentiable weight functions

$$w_\pm(\mathbf{x}) := \frac{H^2}{\pi} \int_{S_\pm} \exp(-\|\mathbf{y} - \mathbf{x}\|^2 H^2) d\mathbf{y}$$

and set

$$W_\pm(x, y) := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w_\pm(u, v) e(-ux - vy) du dv, \quad (8.1)$$

for the corresponding Fourier transform. We collect together some basic properties of these functions in the following result.

Lemma 22. — *Let $N \geq 0$ be an arbitrary integer and let $H \geq 1$. Let $\mathcal{S} \subset \mathbb{R}^2$ be a region enclosed by a piecewise differentiable boundary, which is contained in $[-c, c]^2$ for some $c > 0$. Then the following hold:*

1. *For any $\mathbf{x} \in \mathbb{R}^2$ we have $0 \leq w_\pm(\mathbf{x}) \leq 1$.*
2. *There exists a function $\tilde{w}_- : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that for any $\mathbf{x} \in \mathbb{R}^2$ we have*

$$w_-(\mathbf{x}) + \tilde{w}_-(\mathbf{x}) \leq 1_{\mathcal{S}}(\mathbf{x}) \leq w_+(\mathbf{x}) (1 + O_N(H^{-N})),$$

where $1_{\mathcal{S}}$ is the characteristic function of the set \mathcal{S} and

$$\tilde{w}_-(\mathbf{x}) \ll \frac{e^{-H}}{1 + |\mathbf{x}|^2}.$$

3. We have

$$\int_{\mathbb{R}^2} w_{\pm}(\mathbf{x}) d\mathbf{x} = \text{vol}(S_{\pm}) = \text{vol}(\mathcal{S}) + O\left(\frac{1}{\sqrt{H}}\right).$$

4. We have $W_{\pm}(x, y) \ll_N H^{2N} \max\{|x|, |y|\}^{-N}$.

The implied constants in these estimates depends implicitly on c .

Proof. — To begin with we record the trivial inequalities

$$0 \leq w_{\pm}(\mathbf{x}) \leq \frac{H^2}{\pi} \int_{\mathbb{R}^2} \exp(-\|\mathbf{y} - \mathbf{x}\|^2 H^2) d\mathbf{y} = 1,$$

for any $\mathbf{x} \in \mathbb{R}^2$, which establishes part (1).

Turning to part (2) we observe that for $\mathbf{x} \in \mathcal{S}$ we have

$$\begin{aligned} w_+(\mathbf{x}) &\geq \frac{H^2}{\pi} \int_{\mathbf{y} \in B(\mathbf{x}, \frac{1}{\sqrt{H}})} \exp(-\|\mathbf{y} - \mathbf{x}\|^2 H^2) d\mathbf{y} \\ &= \frac{1}{\pi} \int_{\mathbf{y} \in [-\sqrt{H}, \sqrt{H}]^2} \exp(-\|\mathbf{y}\|^2) d\mathbf{y} \\ &\geq 1 + O(\exp(-H)). \end{aligned}$$

Since $w_+(\mathbf{x}) \geq 0$ when $\mathbf{x} \notin \mathcal{S}$ it easily follows that

$$1_{\mathcal{S}}(\mathbf{x}) \leq w_+(\mathbf{x})(1 + O_N(H^{-N})),$$

for any $N \geq 0$. Next, if $\mathbf{x} \in \mathcal{S}$ then $w_-(\mathbf{x}) \leq 1$, whereas if $\mathbf{x} \notin \mathcal{S}$ then

$$w_-(\mathbf{x}) \leq \frac{H^2}{\pi} \int_{\mathbf{y} \in \mathbb{R}^2 \setminus B(\mathbf{x}, \frac{1}{\sqrt{H}})} \exp(-\|\mathbf{y} - \mathbf{x}\|^2 H^2) d\mathbf{y} = O(\exp(-H)).$$

Noting that $\|\mathbf{y} - \mathbf{x}\|^2 \geq \frac{1}{4}\|\mathbf{x}\|^2$ if $\|\mathbf{y}\| \leq \frac{1}{2}\|\mathbf{x}\|$, we see that $\|\mathbf{y}\| \leq 2c \leq \frac{1}{2}\|\mathbf{x}\|$ if $\|\mathbf{x}\| \geq 4c$ and $\mathbf{y} \in S_{\pm}$, since $\mathcal{S} \subseteq [-c, c]^2$. It follows that

$$w_-(\mathbf{x}) \leq \frac{H^2}{\pi} \exp\left(-\frac{\|\mathbf{x}\|^2 H^2}{4}\right) \int_{\mathbf{y} \in S_+} d\mathbf{y} \ll \exp\left(-\frac{\|\mathbf{x}\|^2 H^2}{5}\right) \ll \frac{\exp(-H)}{|\mathbf{x}|^2},$$

if $\|\mathbf{x}\| \geq 4c$. This therefore suffices to complete the proof of part (2).

Turning to part (3), we note that the estimate for $\text{vol}(S_{\pm})$ is a consequence of the piecewise differentiability of the boundary of \mathcal{S} . Finally part (4) follows from repeated integration by parts. \square

On recalling the definition of the set \mathcal{S}'_u from (6.4), we may clearly write

$$\#(\Lambda \cap \mathcal{R}'(X)) = \#\{(s, t) \in \Lambda : (s/\sqrt{X}, \sqrt{a^2 - b^2}|t|/\sqrt{X}) \in \mathcal{S}'_{b/a}\}.$$

We will use the Poisson summation formula to examine this quantity. First we deduce from Lemma 13 that \mathcal{S}'_u satisfies the hypotheses in Lemma 22. We deduce that

$$\Sigma_- + \tilde{\Sigma}_- \leq \#(\Lambda \cap \mathcal{R}'(X)) \leq \Sigma_+(1 + O_N(H^{-N})),$$

where

$$\Sigma_{\pm} = \sum_{(s,t) \in \Lambda} w_{\pm} \left(\frac{s}{\sqrt{X}}, \frac{\sqrt{|a^2 - b^2|}t}{\sqrt{X}} \right), \quad \tilde{\Sigma}_{-} = \sum_{(s,t) \in \Lambda} \tilde{w}_{-} \left(\frac{s}{\sqrt{X}}, \frac{\sqrt{|a^2 - b^2|}t}{\sqrt{X}} \right). \quad (8.2)$$

We will begin by examining the contribution from the sums Σ_{\pm} .

Let us recall from (6.1) that

$$\Lambda = \{(s, t) \in \mathbb{Z}^2 : [\lambda'_1, \ell] \mid s, \lambda'_2 \mid s - at \text{ and } \ell \mid t\},$$

for odd positive integers λ'_i, ℓ such that $\lambda'_i \mid a^2 + (-1)^i b^2$ and $\lambda'_i = k_i \lambda_i$. In particular $\gcd(\lambda'_1, \lambda'_2) = \gcd(\lambda'_i, ab) = 1$. We will find it convenient to set $d_i := \gcd(\ell, \lambda'_i)$ and $\lambda''_i := \lambda'_i / d_i$, for $i = 1, 2$. For any $(s, t) \in \Lambda$ we may therefore make the change of variables

$$s = \ell \lambda''_1 \sigma, \quad s - at = \ell \lambda''_2 \tau,$$

Recalling that $\ell \mid t$, we see that under this change of variables we have

$$\lambda''_1 \sigma \equiv \lambda''_2 \tau \pmod{a}, \quad (8.3)$$

and there is clearly a bijection between elements of Λ and solutions to this congruence. We therefore have

$$\Sigma_{\pm} = \sum_{\substack{(\sigma, \tau) \in \mathbb{Z}^2 \\ (8.3) \text{ holds}}} w_{\pm} \left(\frac{\ell \lambda''_1 \sigma}{\sqrt{X}}, \frac{\sqrt{|a^2 - b^2|}(\ell \lambda''_1 \sigma - \ell \lambda''_2 \tau)}{a \sqrt{X}} \right).$$

say. Breaking the sum into residue classes modulo a , an application of the Poisson summation formula yields

$$\begin{aligned} \Sigma_{\pm} &= \sum_{\substack{(\alpha, \beta) \pmod{a} \\ \lambda''_1 \alpha \equiv \lambda''_2 \beta \pmod{a}}} \sum_{\substack{(\sigma, \tau) \in \mathbb{Z}^2 \\ \sigma \equiv \alpha \pmod{a} \\ \tau \equiv \beta \pmod{a}}} w_{\pm} \left(\frac{\ell \lambda''_1 \sigma}{\sqrt{X}}, \frac{\sqrt{|a^2 - b^2|}(\ell \lambda''_1 \sigma - \ell \lambda''_2 \tau)}{a \sqrt{X}} \right) \\ &= \frac{X}{a^2 \ell^2 \lambda''_1 \lambda''_2} \sum_{\substack{(\alpha, \beta) \pmod{a} \\ \lambda''_1 \alpha \equiv \lambda''_2 \beta \pmod{a}}} \sum_{(m, n) \in \mathbb{Z}^2} e \left(\frac{\alpha m + \beta n}{a} \right) W_{\pm, b/a} \left(\frac{m}{\lambda''_1 T}, \frac{n}{\lambda''_2 T} \right), \end{aligned}$$

where

$$T = \frac{a \ell}{\sqrt{X}}, \quad (8.4)$$

and for $\delta > 0$ we have temporarily set

$$W_{\pm, \delta}(x, y) := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w_{\pm}(u, \sqrt{|1 - \delta^2|}(u - v)) e(-ux - vy) \, du \, dv.$$

It is easy to see that $W_{\pm, \delta}(x, y) = W_{\pm}(x + y, y / \sqrt{|1 - \delta^2|}) / \sqrt{|1 - \delta^2|}$ in the notation of (8.1). We may therefore write

$$\Sigma_{\pm} = \frac{Z}{a} \sum_{\substack{(\alpha, \beta) \pmod{a} \\ \lambda''_1 \alpha \equiv \lambda''_2 \beta \pmod{a}}} \sum_{(m, n) \in \mathbb{Z}^2} e \left(\frac{\alpha m + \beta n}{a} \right) W_{\pm} \left(\frac{m / \lambda''_1 + n / \lambda''_2}{T}, \frac{an}{\sqrt{|a^2 - b^2|} \lambda''_2 T} \right),$$

where

$$Z := \frac{X}{\ell^2 \lambda_1'' \lambda_2'' \sqrt{|a^2 - b^2|}} = \frac{X}{\det \Lambda \sqrt{|a^2 - b^2|}},$$

by Lemma 12.

It follows that $\Sigma_{\pm} = \mathcal{M}_{\pm} + \mathcal{E}_{\pm}$, where $\mathcal{M}_{\pm} = ZW_{\pm}(0, 0)$ and \mathcal{E}_{\pm} is the overall contribution from non-zero vectors in the summation over (m, n) . Now it is clear from Lemmas 13 and 22 that

$$\mathcal{M}_{\pm} = Z \int_{\mathbb{R}^2} w_{\pm}(\mathbf{x}) \, d\mathbf{x} = Z \operatorname{vol}(\mathcal{S}'_{b/a}) + O\left(\frac{Z}{\sqrt{H}}\right) = \frac{\operatorname{vol}(\mathcal{R}'(X))}{\det \Lambda} + O\left(\frac{Z}{\sqrt{H}}\right).$$

Recalling the definition (6.6) of X and the expression for $\det \Lambda$ in Lemma 12, we see that the error term here contributes

$$\begin{aligned} &\ll \frac{B}{\sqrt{H}} \sum_{(a,b) \in \mathcal{A}_1} \sum_{\nu} \sum_{\substack{k_1 \lambda_1 | a^2 - b^2 \\ k_1 \leq K}} \sum_{\substack{k_2 \lambda_2 | a^2 + b^2 \\ k_2 \leq K}} \sum_{\substack{\ell \leq Z_1^{c/10}}} \frac{\gcd(\ell, k_1 k_2 \lambda_1 \lambda_2) 2^{\nu}}{\ell^2 k_1 k_2 \max\{a, b\} \sqrt{|a^2 - b^2|}} \\ &\ll \frac{B(\log B)^2 \log \log B}{\sqrt{H}} \sum_{(a,b) \in \mathcal{A}_1} \frac{\tau(|a^4 - b^4|)}{\max\{a, b\}^{3/2} |a - b|^{1/2}} \\ &\ll \frac{B(\log B)^6 \log \log B}{\sqrt{H}}, \end{aligned}$$

by the proof of (4.3), once summed over all of the remaining parameters. This is satisfactory for Lemma 21 if $H \geq (\log B)^6$, which we now assume.

Let us turn to the estimation of \mathcal{E}_{\pm} , observing that

$$\sum_{\substack{(\alpha, \beta) \bmod a \\ \lambda_1'' \alpha \equiv \lambda_2'' \beta \bmod a}} e\left(\frac{\alpha m + \beta n}{a}\right) = \begin{cases} a, & \text{if } n\lambda_1'' + m\lambda_2'' \equiv 0 \bmod a, \\ 0, & \text{otherwise.} \end{cases}$$

We may therefore write

$$\mathcal{E}_{\pm} = Z \sum_{\substack{(m,n) \in \mathbb{Z}^2 \setminus \{0\} \\ n\lambda_1'' + m\lambda_2'' \equiv 0 \bmod a}} W_{\pm}\left(\frac{m/\lambda_1'' + n/\lambda_2''}{T}, \frac{an}{\sqrt{|a^2 - b^2|} \lambda_2'' T}\right),$$

where T is given by (8.4). We begin by considering the overall contribution from terms with $m = 0$ in this summation. Recalling that $|a - b| \geq Z_2^{-2} \max\{a, b\}$ for any a, b under consideration, we deduce from an application of part (4) of Lemma 22 with $N = 1$ that this part of the sum is

$$\begin{aligned} Z \sum_{n' \in \mathbb{Z} \setminus \{0\}} W_{\pm}\left(\frac{2an'}{\lambda_2'' T}, \frac{2a^2 n'}{\sqrt{|a^2 - b^2|} \lambda_2'' T}\right) &\ll \frac{H^2 \lambda_2'' Z T \log B}{a} \\ &\ll \frac{\gcd(\ell, k_1 \lambda_1)}{\ell} \cdot \sqrt{\frac{\lambda_2}{\lambda_1}} \cdot \frac{H^2 \sqrt{B} (\log B)^{1+\varepsilon}}{k_1 \max\{a, b\}^{3/2}}. \end{aligned}$$

Recalling the statement of Lemma 17 it follows from the definition of (7.3) that

$$\frac{k_2 \lambda_2}{k_1 \lambda_1} \leq \frac{B}{K \max\{a, b\}}.$$

Hence the above sum is

$$\ll \frac{\gcd(\ell, k_1 \lambda_1)}{\ell} \cdot \frac{H^2 B (\log B)^{1+\varepsilon}}{\sqrt{k_1 k_2} \max\{a, b\}^2 \sqrt{K}}.$$

Once inserted into Lemma 17, this makes the overall contribution

$$\begin{aligned} &\ll \frac{H^2 B (\log B)^2}{\sqrt{K}} \sum_{(a,b) \in \mathcal{A}_1} \sum_{\substack{k_1 \lambda_1 | a^2 - b^2 \\ k_1 \leq K}} \sum_{\substack{k_2 \lambda_2 | a^2 + b^2 \\ k_2 \leq K}} \frac{\tau(k_1 \lambda_1)}{\sqrt{k_1 k_2} \max\{a, b\}^2} \\ &\ll \frac{H^2 B (\log B)^2}{\sqrt{K}} \sum_{(a,b) \in \mathcal{A}_1} \frac{\tau(a^2 - b^2) \tau_3(a^2 - b^2) \tau_3(a^2 + b^2)}{\max\{a, b\}^2}, \end{aligned}$$

since $\sum_{\ell \leq L} \gcd(\ell, a) \ell^{-1} \ll \tau(a) \log L$ for any non-zero integer a . Combining Lemma 4 with a dyadic summation gives the contribution

$$\ll \frac{H^2 B Z_2^2 (\log B)^{11}}{\sqrt{K}},$$

which in view of (6.5) is satisfactory for Lemma 21 if $H \leq (\log B)^{17/2}$. Similarly, the overall contribution from terms with $n = 0$ is seen to be satisfactory. In view of our earlier constraints on H we are led to take the value

$$H = (\log B)^6$$

in our construction of the weight functions w_{\pm} .

Next we consider the contribution from terms with $m \lambda_2'' + n \lambda_1'' = 0$. The general solution of this equation is $(m, n) = k(-\lambda_1'', \lambda_2'')$ for non-zero integer k . We therefore use part (4) of Lemma 22 to derive the contribution

$$\begin{aligned} Z \sum_{k \in \mathbb{Z} \setminus \{0\}} W_{\pm} \left(0, \frac{ak}{\sqrt{|a^2 - b^2|} T} \right) &\ll \frac{H^2 \sqrt{|a^2 - b^2|} Z T \log B}{a} \\ &\ll \frac{\gcd(\ell, k_1 \lambda_1 k_2 \lambda_2)}{\ell} \cdot \frac{1}{\sqrt{\lambda_1 \lambda_2}} \cdot \frac{H^2 \sqrt{B} \log B}{k_1 k_2 \max\{a, b\}^{1/2}} \end{aligned}$$

from this part of the sum. It follows from (7.3) that

$$\frac{1}{k_1 \lambda_1 k_2 \lambda_2} \leq \frac{B}{K \max\{a, b\}^3}.$$

Hence the above sum is

$$\ll \frac{\gcd(\ell, k_1 \lambda_1 k_2 \lambda_2)}{\ell} \cdot \frac{H^2 B \log B}{\sqrt{k_1 k_2} \max\{a, b\}^2 \sqrt{K}},$$

which leads to a satisfactory contribution to Lemma 17 as argued for the case $m = 0$. Similarly, one deduces that there is a satisfactory overall contribution to the sum from vectors (m, n) for which $(a^2 - b^2)m/\lambda_1'' = (a^2 + b^2)n/\lambda_2''$.

In what follows we may therefore approximate \mathcal{E}_\pm by the corresponding sum \mathcal{E}'_\pm , say, in which

$$mn \neq 0, \quad m\lambda_2'' + n\lambda_1'' \neq 0, \quad \frac{(a^2 - b^2)m}{\lambda_1''} \neq \frac{(a^2 + b^2)n}{\lambda_2''},$$

with a satisfactory error. Now it is clear that $n\lambda_1'' + m\lambda_2'' \equiv 0 \pmod{a}$ if and only if $n\ell\lambda_1'' + m\ell\lambda_2'' = alt'$ for some integer t' . Let us write $s = \ell\lambda_1''n$ and $t = \ell t'$. Then we have

$$\mathcal{E}'_\pm = Z \sum_{\substack{(s,t) \in \Lambda \\ st(s-at) \neq 0 \\ s/t \neq (a^2 - b^2)/(2a)}} W_\pm \left(\frac{at}{\ell\lambda_1''\lambda_2''T}, \frac{as}{\sqrt{|a^2 - b^2|}\ell\lambda_1''\lambda_2''T} \right).$$

We may freely assume that $|s|, |t| \leq B^{3/2}$ in this summation.

Recall the definition of the quadratic forms $Q_j(s, t)$ from (5.1) for $1 \leq j \leq 3$. Arguing similarly to above it is now possible to show that there is a negligible contribution from (s, t) for which one of these forms vanishes. To illustrate the argument let us consider the overall contribution from (s, t) for which $Q_2(s, t) = 0$. But then there exists an integer c for which $a^2 - b^2 = 2c^2$ and $s/t = \pm 2c$. Thus we have $(s, t) = k(\pm 2c, 1)$ for non-zero integer k and the contribution to \mathcal{E}'_\pm is

$$\ll \frac{H^2 \ell \lambda_1'' \lambda_2'' Z T}{a} \sum_{\substack{k \in \mathbb{Z} \setminus \{0\} \\ \ell \lambda_1'' | 2kc, \ell \lambda_2'' | k(\pm 2c - a)}} \frac{1}{k}.$$

One notes that $\gcd(\lambda_2'', \pm 2c - a) \leq 3$ since $\lambda_2'' \mid a^2 + b^2$ and $\pm 2c - a$ divides $a^2 - 4c^2 = 2b^2 - a^2$. Hence this is

$$\begin{aligned} &\ll \frac{H^2 \lambda_1'' Z T \log B}{a} \ll \frac{\gcd(\ell, k_1 \lambda_1)}{\ell} \cdot \sqrt{\frac{k_1 \lambda_1}{k_2 \lambda_2}} \cdot \frac{H^2 \sqrt{B} (\log B)^{1+\varepsilon}}{\max\{a, b\}^{3/2}} \\ &\ll \frac{\gcd(\ell, k_1 \lambda_1)}{\ell} \cdot \frac{H^2 B (\log B)^{1+\varepsilon}}{\max\{a, b\}^2 \sqrt{K}}, \end{aligned}$$

by (7.3). This makes a satisfactory overall contribution as previously. We may therefore approximate \mathcal{E}'_\pm by

$$\mathcal{E}''_\pm = Z \sum_{\substack{(s,t) \in \Lambda \\ st(s-at) \neq 0, Q_j(s,t) \neq 0 \\ s/t \neq (a^2 - b^2)/(2a)}} W_\pm \left(\frac{at}{\ell\lambda_1''\lambda_2''T}, \frac{as}{\sqrt{|a^2 - b^2|}\ell\lambda_1''\lambda_2''T} \right) = \mathcal{E}''_{\pm,1} + \mathcal{E}''_{\pm,2},$$

with satisfactory error, where $\mathcal{E}''_{\pm,1}$ is the contribution from s, t such that

$$\sqrt{|a^2 - b^2|}|t| > |s|$$

and $\mathcal{E}''_{\pm,2}$ is the corresponding contribution from the opposite inequality. We will deal here with the sum $\mathcal{E}''_{\pm,1}$, the remaining sum being handled along similar lines. Taking

$N = 2$ in part (4) of Lemma 22 we observe that

$$\mathcal{E}_{\pm,1}'' \ll \frac{H^4 Z(\ell \lambda_1'' \lambda_2'' T)^2}{a^2} \sum_{\substack{(s,t) \in \Lambda \\ st(s-at) \neq 0, Q_j(s,t) \neq 0 \\ s/t \neq (a^2-b^2)/(2a) \\ \sqrt{|a^2-b^2|}|t| > |s|}} \frac{1}{t^2}. \quad (8.5)$$

Let $R, A, L_1, L_2 > 0$. We will estimate the overall contribution from $\mathcal{E}_{\pm,1}''$ once inserted into Lemma 17, with

$$A \leq \max\{a, b\} < 2A, \quad L_1 \leq \lambda_1 < 2L_1, \quad L_2 \leq \lambda_2 < 2L_2, \quad R < |t| \leq 2R.$$

We call this contribution $E = E_{\pm}(R, A, L_1, L_2)$, say, and we may clearly assume that

$$R, A, L_1, L_2 \gg 1, \quad R \leq B^{3/2}, \quad A \leq \sqrt{B}/Z_1^c, \quad L_i \leq 2A^2,$$

for $i = 1, 2$. We will show that

$$E \ll \frac{BH^4 Z_1^{O(1)}}{Z_1^c}. \quad (8.6)$$

Once summed over the $O((\log B)^4)$ dyadic ranges for R, A, L_1, L_2 , this will clearly suffice to complete our treatment of the error term in Lemma 21 since $H = (\log B)^6$.

Recall from (4.4) that $\min\{a, b, |a-b|\} \geq \max\{a, b\}/Z_2^2$, so that

$$\frac{A}{Z_2^2} \leq \min\{a, b, |a-b|\} \leq 2A.$$

Furthermore we have $k_1, k_2 \leq K$. Thus we see that the contribution to our estimate for $\mathcal{E}_{\pm,1}''$ from t, λ_i and a restricted to lie in the above dyadic intervals is

$$\begin{aligned} & \ll \frac{H^4 (\log B)^\varepsilon Z(\ell \lambda_1'' \lambda_2'' T)^2}{a^2 R^2} \# \left\{ (s, t) \in \Lambda : \begin{array}{l} Q_j(s, t) \neq 0 \text{ for } 1 \leq j \leq 3, \\ st(s-at) \neq 0, \\ s/t \neq (a^2-b^2)/(2a), \\ |s| \leq 2\sqrt{|a^2-b^2|}R, \quad |t| \leq 2R \end{array} \right\} \\ & \ll \frac{H^4 \ell^2 K^2 L_1 L_2 (\log B)^\varepsilon}{AR^2} \# (\Lambda \cap \mathcal{R}^\dagger(cA^2 R^2)), \end{aligned}$$

for some absolute constant $c > 0$, in the notation of (5.8). Recalling that $\ell \leq Z_1^{c/10}$ it now follows that

$$E \ll \frac{H^4 L_1 L_2 Z_1^{c/5+O(1)}}{AR^2} \sum_{\substack{(a,b) \in \mathbb{N}^2 \\ \gcd(a,b)=1, ab \neq 1 \\ \max\{a,b\} \leq 2A}} \sum_{\substack{k_i \lambda_i | a^2 + (-1)^i b^2 \\ k_i \leq K}} \sum_{\ell \in \mathbb{N}} \# (\Lambda \cap \mathcal{R}^\dagger(cA^2 R^2)),$$

To analyse this sum we note that $\# (\Lambda \cap \mathcal{R}^\dagger(cA^2 R^2)) \leq L_{k_1, k_2, \ell}^\dagger(Y, 1/2)$, with

$$\frac{A^3 R^2}{L_1 L_2} \ll Y \ll \frac{A^3 R^2}{L_1 L_2}.$$

One observes that $Y \gg 1$, since $\lambda_1 \leq |s| \ll AR$ and $\lambda_2 \leq |s - at| \ll AR$, whence $L_1 L_2 \ll A^2 R^2$ in order for the summand not to vanish. It is trivial to see that $Y \ll B^5$. It therefore follows from Lemma 15 that

$$E \ll \frac{H^4 L_1 L_2 Z_1^{c/5+O(1)}}{AR^2} \cdot Y \ll H^4 A^2 Z_1^{c/5+O(1)} \ll \frac{BH^4 Z_1^{O(1)}}{Z_1^{9c/5}},$$

which thereby establishes (8.6).

Our final task is to deal with the contribution from the sum $\tilde{\Sigma}_-$ in (8.2). Focusing without loss of generality on the contribution from $\sqrt{|a^2 - b^2|}|t| > |s|$, we may apply the estimate for \tilde{w}_- in Lemma 22 to deduce that

$$\tilde{\Sigma}_- \ll \frac{e^{-H} X}{|a^2 - b^2|} \sum_{\substack{(s,t) \in \Lambda \\ \sqrt{|a^2 - b^2|}|t| > |s|}} \frac{1}{t^2}.$$

Arguing as before we may readily restrict attention to s, t for which $st(s - at) \neq 0$, $Q_j(s, t) \neq 0$ and $s/t \neq (a^2 - b^2)/(2a)$. But then we are led to an upper bound which matches the estimate for $\mathcal{E}_{\pm,1}''$ presented in (8.5), but with different factors in front of the sum. Mimicking the ensuing argument based on dyadic intervals and recalling the definition X of (6.6), one finds that our analogue of $E = E_-(R, A, L_1, L_2)$ is bounded by

$$\ll \frac{e^{-H} Z_1^{c/5+O(1)} L_1 L_2 B}{A^3 R^2} \cdot Y \ll e^{-H} Z_1^{c/5+O(1)} B.$$

This is satisfactory with our choice of $H = (\log B)^6$, which thereby completes the proof of Lemma 21.

9. The main term

We now draw together the various main terms that appear in Lemmas 18, 19 and 20, and insert them into Lemma 17's estimate for $N_1(B)$. Recall the definition (7.12) of π_{ab} . We may clearly bring the summation over ν to the innermost sum, finding that

$$\begin{aligned} \sum_{\nu \leq \max\{1, \nu_2(a^2 - b^2)\}} L^\nu(B) &= \left(\pi_{ab} \left(\frac{1}{2} - \frac{1}{2^{[2Z_2]}} \right) + 1 - \frac{\pi_{ab}}{2} + \pi_{ab+1}(\nu_2(a^2 - b^2) - 3) \right. \\ &\quad \left. + \pi_{ab+1} \left(2 - \frac{2^{\nu_2(a^2 - b^2)}}{2^{[2Z_2]+2}} \right) \right) \frac{\text{vol}(\mathcal{R}'_0)}{\det \Lambda} \\ &= (\delta_{a,b} + O(2^{-Z_2})) \frac{\text{vol}(\mathcal{R}'_0)}{\det \Lambda}, \end{aligned}$$

where if $g = h * \tau = h * 1 * 1$ is given by (2.6) and (2.7), then

$$\delta_{a,b} := \max\{1, \nu_2(a^2 - b^2)\} = g(2^{\nu_2(a^4 - b^4)}).$$

Here we have equated $L^\nu(B) = \tilde{L}_{k_1, k_2, \ell}(B)$ with the main term in the various estimates from §7. Moreover, we have used the fact that $\nu_2(a^2 - b^2) \leq Z_2$ in controlling the error term. We may therefore deduce from Lemma 13 that

$$\sum_{\nu \leq \max\{1, \nu_2(a^2 - b^2)\}} L^\nu(B) = \left(\delta_{a,b} + O\left(\frac{1}{\log B}\right) \right) \frac{\lambda_1 \lambda_2 B \operatorname{vol}(\mathcal{S}_{b/a})}{(\det \Lambda) \max\{a, b\} \sqrt{|a^2 - b^2|}},$$

where \mathcal{S}_u is given by (6.3) for any positive $u \neq 1$.

Define

$$f(u) := \frac{\operatorname{vol}(\mathcal{S}_u) + \operatorname{vol}(\mathcal{S}_{1/u})}{\sqrt{1 - u^2}}, \quad (9.1)$$

for any $u \in (0, 1)$. Let $\chi_{t_1, t_2}(R)$ be the characteristic function of the region (7.3) and let

$$h(a, b; Y) := \sum_{\substack{n|a^4 - b^4 \\ d_i = \gcd(n, a + (-1)^i b) \\ d_3 = \gcd(n, a^2 + b^2)}} (1 * h)(n) \chi_{d_1 d_2, d_3}(Y), \quad (9.2)$$

for any $Y \geq 1$. We are now ready to establish the following result.

Lemma 23. — *We have*

$$\begin{aligned} N_1(B) &\leq \frac{8B}{3\zeta(2)} \sum_{(a,b) \in \mathcal{A}_2} \frac{f(b/a) h(a, b; B 2^{Z_2+1}/K)}{a^2} + O\left(\frac{B(\log B)^4}{\log \log B}\right) \\ N_1(B) &\geq \frac{8B}{3\zeta(2)} \sum_{(a,b) \in \mathcal{A}_2} \frac{f(b/a) h(a, b; B/K)}{a^2} + O\left(\frac{B(\log B)^4}{\log \log B}\right), \end{aligned}$$

where $h(a, b; Y)$ is given by (9.2) and

$$\mathcal{A}_2 := \left\{ (a, b) \in \mathbb{N}^2 : a/Z_2^2 \leq b \leq a(1 - 1/Z_2^2), \ a < \sqrt{B}, \ \gcd(a, b) = 1 \right\}.$$

Proof. — In order to avoid repeating the same argument twice we will simply replace $\delta_{a,b} + O(1/\log B)$ by $\delta_{a,b}$ in what follows, leaving it to the patience of the reader to verify that this is indeed permissible. Bringing together our expression for $\sum_\nu L^\nu(B)$ with Lemmas 12, 17 and 21, we deduce that $N_1(B)$ can be replaced by

$$2B \sum_{(a,b) \in \mathcal{A}_1} \delta_{a,b} F(a, b) \sum_{\substack{k_1 \lambda_1 | a^2 - b^2 \\ k_1 \leq K}}^b \sum_{\substack{k_2 \lambda_2 | a^2 + b^2 \\ k_2 \leq K}}^b \chi_{k_1 \lambda_1, k_2 \lambda_2} \left(\frac{B}{K} \right) \frac{\mu(k_1) \mu(k_2)}{k_1 k_2} \sigma(Z_1^{c/10}),$$

with an acceptable error, where K is given by (6.5), $\chi_{t_1, t_2}(R)$ is the characteristic function of (7.3), $F(a, b) := \operatorname{vol}(\mathcal{S}_{b/a}) / (\max\{a, b\} \sqrt{|a^2 - b^2|})$ and

$$\sigma(T) := \sum_{\ell \leq T}^b \mu(\ell) \frac{\gcd(k_1 k_2 \lambda_1 \lambda_2, \ell)}{\ell^2} = \frac{4}{3\zeta(2) \varphi^\dagger(k_1 k_2 \lambda_1 \lambda_2)} + O\left(\frac{\tau(k_1 k_2 \lambda_1 \lambda_2)}{T}\right),$$

for any $T \geq 1$. Lemma 13 implies that $\operatorname{vol}(\mathcal{S}_{b/a}) \ll 1$ and the definition (4.4) of \mathcal{A}_1 yields $\sqrt{|a^2 - b^2|} \geq \max\{a, b\}/Z_2$. Hence the overall contribution from the above error term is clearly satisfactory. Applying Lemma 4 one easily checks that the

overall contribution to $N_1(B)$ from values of k_1, k_2 such that $\max\{k_1, k_2\} > K$ is also satisfactory.

We may now conclude that

$$N_1(B) = \frac{8B}{3\zeta(2)} \sum_{(a,b) \in \mathcal{A}_1} F(a,b) h_0(a,b; B) + O\left(\frac{B(\log B)^4}{\log \log B}\right),$$

where

$$h_0(a,b; B) := \delta_{a,b} \sum_{k_1 \lambda_1 | a^2 - b^2}^{\flat} \frac{\mu(k_1)}{k_1 \varphi^\dagger(k_1 \lambda_1)} \sum_{k_2 \lambda_2 | a^2 + b^2}^{\flat} \frac{\mu(k_2)}{k_2 \varphi^\dagger(k_2 \lambda_2)} \chi_{k_1 \lambda_1, k_2 \lambda_2} \left(\frac{B}{K}\right).$$

Recall the definition of φ^* from (1.10). For any arithmetic function f , we have

$$\sum_{k_i \lambda_i | N}^{\flat} \frac{\mu(k_i)}{k_i} f(k_i \lambda_i) = \sum_{n | N}^{\flat} f(n) \sum_{k_i | n} \frac{\mu(k_i)}{k_i} = \sum_{n | N}^{\flat} \varphi^*(n) f(n).$$

It therefore follows that

$$\begin{aligned} h_0(a,b; B) &= \delta_{a,b} \sum_{m | a^2 - b^2}^{\flat} \frac{\varphi^*(m)}{\varphi^\dagger(m)} \sum_{m_3 | a^2 + b^2}^{\flat} \frac{\varphi^*(m_3)}{\varphi^\dagger(m_3)} \chi_{m, m_3} \left(\frac{B}{K}\right) \\ &= \delta_{a,b} \sum_{\substack{n | a^4 - b^4 \\ m = (n, a^2 - b^2)_{\flat} \\ m_3 = (n, a^2 + b^2)_{\flat}}} (1 * h)(mm_3) \chi_{m, m_3} \left(\frac{B}{K}\right) \\ &= \sum_{\substack{n | a^4 - b^4 \\ m_i = (n, a + (-1)^i b)_{\flat} \\ m_3 = (n, a^2 + b^2)_{\flat}}} (1 * h)(n) \chi_{m_1 m_2, m_3} \left(\frac{B}{K}\right), \end{aligned}$$

where h is given by (2.7). Recall the inequality $\nu_2(a^2 - b^2) \leq Z_2$ satisfied by any $(a, b) \in \mathcal{A}_1$. A little thought reveals that

$$h(a,b; B/K) \leq h_0(a,b; B) \leq h(a,b; B2^{Z_2+1}/K),$$

in the notation of (9.2).

Bringing everything together we have so far established the upper and lower bounds

$$\begin{aligned} N_1(B) &\leq \frac{8B}{3\zeta(2)} \sum_{(a,b) \in \mathcal{A}_1} F(a,b) h(a,b; B2^{Z_2+1}/K) + O\left(\frac{B(\log B)^4}{\log \log B}\right) \\ N_1(B) &\geq \frac{8B}{3\zeta(2)} \sum_{(a,b) \in \mathcal{A}_1} F(a,b) h(a,b; B/K) + O\left(\frac{B(\log B)^4}{\log \log B}\right), \end{aligned}$$

We proceed to enlarge the set of allowable a, b slightly, by handling separately the contribution from a, b such that $\nu_2(a^2 - b^2) > Z_2$ or $\max\{a, b\} > Z_2^2 |a - b|$. For the former, exactly the same sort of argument employed in Lemma 9 suffices to show that

we obtain an overall contribution

$$\ll \frac{B}{Z_2} \sum_{a,b < \sqrt{B}/Z_1^c} \frac{\nu_2(a^2 - b^2)\tau(|a^4 - b^4|)}{\max\{a, b\}^{3/2}|a - b|^{1/2}} \ll \frac{B(\log B)^4}{\log \log B},$$

which is satisfactory. For the latter, the proof of (4.3) shows that there is a satisfactory contribution in this case too. Arguing in a similar manner it is also possible to restrict attention to a, b for which $\min\{a, b\} \leq \max\{a, b\}(1 - 1/Z_2^2)$ and to enlarge the set of allowable a, b to include the range $\sqrt{B}/Z_1^c \leq \max\{a, b\} < \sqrt{B}$.

Finally, we break the summation over a, b into those for which $a > b$ and those for which $a < b$, observing that

$$F(a, b) + F(b, a) = \frac{\text{vol}(\mathcal{S}_{b/a}) + \text{vol}(\mathcal{S}_{a/b})}{\max\{a, b\}\sqrt{|a^2 - b^2|}}.$$

This therefore allows us to restrict to a summation over the set \mathcal{A}_2 , which thereby completes the proof of the lemma. \square

We now have everything in place to complete the proof of the theorem. In what follows let us write Y for either of the quantities $B2^{Z_2+1}/K$ or B/K . Define

$$\Sigma_{\theta_1, \theta_2}(Y) := \sum_{\substack{\theta_1 a \leq b \leq (1-\theta_2)a \\ a < \sqrt{B}, \gcd(a, b)=1}} \frac{h(a, b; Y)}{a^2},$$

for any $\theta_1, \theta_2 > 0$ such that $\theta_1 + \theta_2 < 1$. Observing that $f(u) = f(0) + \int_0^u f'(t) dt$, it now follows that

$$\sum_{(a, b) \in \mathcal{A}_2} \frac{f(b/a)h(a, b; Y)}{a^2} = f(0)\Sigma_{Z_2^{-2}, Z_2^{-2}}(Y) + \int_0^{1-Z_2^{-2}} f'(t)\Sigma_{\max\{t, Z_2^{-2}\}, Z_2^{-2}}(Y) dt. \quad (9.3)$$

Hence Lemma 23 renders it sufficient to estimate $\Sigma_{\theta_1, \theta_2}(Y)$ asymptotically. This is achieved in the following result.

Lemma 24. — *We have*

$$\Sigma_{\theta_1, \theta_2}(Y) = \frac{C^*(1 - \theta_1 - \theta_2)}{36}(\log B)^4 + O((\log B)^{3+\varepsilon}),$$

where C^* is given by (2.4), with $(L_1, L_2, Q) = (x_1 - x_2, x_1 + x_2, x_1^2 + x_2^2)$ and g given by (2.6).

Proof. — Recall the definition (9.2) of $h(a, b; Y)$ and define the region

$$W := \left\{ \mathbf{w} \in \mathbb{R}_{\geq 0}^4 : \begin{array}{l} w_1 + w_2 + w_4 \leq 1 + 2w_3, \\ 2w_3 + w_4 \leq 1 + w_1 + w_2, \\ 3w_4 \leq 1 + w_1 + w_2 + 2w_3, \\ w_1 + w_2 + 2w_3 \leq 1 + w_4 \end{array} \right\}.$$

This is contained in $[0, 1]^4$. On recalling the definition (7.3) of $V_{a,b}(R)$ for $(a, b) \in \mathcal{A}_2$, one easily checks that

$$h(a, b; Y) = \sum_{\substack{n|a^4-b^4 \\ d_i=\gcd(n, a+(-1)^i b) \\ d_3=\gcd(n, a^2+b^2) \\ (\frac{\log d_1}{\log Y}, \frac{\log d_2}{\log Y}, \frac{\log d_3}{2\log Y}, \frac{\log a}{\log Y}) \in W}} (1 * h)(n).$$

It now follows from Lemma 5 that

$$\Sigma_{\theta_1, \theta_2}(Y) = 2C^*(1 - \theta_1 - \theta_2) \text{vol}(W_0)(\log B)^4 + O((\log B)^{3+\varepsilon}),$$

where W_0 is given by (2.5). Here we have applied the lemma with the region

$$\mathcal{B} = \{\mathbf{x} \in (0, 1)^2 : \theta_1 x_1 \leq x_2 \leq (1 - \theta_2)x_1\},$$

which has volume $(1 - \theta_1 - \theta_2)/2$. We can calculate the volume of W_0 using the software package `polymake` [15], the outcome being that $\text{vol}(W_0) = 1/72$. The statement of the lemma is now obvious. \square

Recalling the definition of the function (9.1), it is a routine calculation to verify that

$$f(u) \ll 1 + \log(1 - u), \quad f'(u) \ll (1 - u)^{-1}$$

for $u \in (0, 1)$, using Lemma 13 and the definition (6.3) of \mathcal{S}_u . In particular it follows that $\int_0^{1-1/Z_2^2} |f'(t)| dt \ll \log Z_2 \ll \log \log \log B$. Combining this with Lemma 24 and (9.3) in Lemma 23, and inserting it into Lemma 6, we arrive at the final bound

$$N_{U,H}(B) = CB(\log B)^4 + O\left(\frac{B(\log B)^4}{\log \log B}\right),$$

where

$$C = \frac{16C^*}{27\zeta(2)} \left(f(0) + \int_0^1 (1 - u) f'(u) du \right) = \frac{16C^*}{27\zeta(2)} \int_0^1 f(u) du.$$

Finally, we note that

$$\begin{aligned} \int_0^1 f(u) du &= \int_0^1 \int_{\mathcal{S}_u} \frac{1}{\sqrt{1 - u^2}} ds dt du + \int_0^1 \int_{\mathcal{S}_{1/u}} \frac{1}{\sqrt{1 - u^2}} ds dt du \\ &= \int_0^1 \int_{\mathcal{S}_u} \frac{1}{\sqrt{1 - u^2}} ds dt du + \int_1^\infty \int_{\mathcal{S}_u} \frac{1}{u\sqrt{u^2 - 1}} ds dt du \\ &= \sigma_\infty, \end{aligned}$$

where

$$\sigma_\infty := \int \int \int_{\left\{ \begin{array}{l} 0 < \max\{1, u\} p_u(s, t) \leq 1, \\ 0 < \max\{1, u\} q_u(s, t) \leq 1, \\ t, u > 0, \quad r_u(s, t) > 0 \end{array} \right\}} \frac{1}{\sqrt{|1 - u^2|}} ds dt du. \quad (9.4)$$

This therefore completes the proof of the asymptotic formula

$$N_{U,H}(B) = \frac{16C^*\sigma_\infty}{27\zeta(2)} B(\log B)^4 + O\left(\frac{B(\log B)^4}{\log \log B}\right), \quad (9.5)$$

where C^* is given by (2.4).

10. Peyre's constant

The purpose of this section is to affirm that the constant in our asymptotic formula (9.5) for $N_{U,H}(B)$ agrees with the Peyre's prediction [22], as required to complete the proof of the theorem. In general terms the constant $c_{X,H}$ should be a product of three constants $\alpha(X), \beta(X), \tau_H(X)$.

The constant $\alpha(X)$ is a rational number defined in terms of the cone of effective divisors of X . To calculate its value we will apply the calculations undertaken by Derenthal in the appendix to this paper. Let $\varepsilon_1, \varepsilon_2 \in \{-1, +1\}$ and let $i = \sqrt{-1}$. Then a straightforward calculation reveals that the 16 lines on X are given by

$$\begin{aligned} M_1(\varepsilon_1, \varepsilon_2): \quad & \begin{cases} x_0 = \varepsilon_1 x_2 = \varepsilon_2 x_4, \\ x_1 = \varepsilon_1 x_3, \end{cases} & M_2(\varepsilon_1, \varepsilon_2): \quad & \begin{cases} x_1 = \varepsilon_1 x_2 = \varepsilon_2 x_4, \\ x_0 = \varepsilon_1 x_3, \end{cases} \\ M_3(\varepsilon_1, \varepsilon_2): \quad & \begin{cases} x_3 = \varepsilon_1 i x_0 = \varepsilon_2 i x_4, \\ x_1 = \varepsilon_1 i x_2, \end{cases} & M_4(\varepsilon_1, \varepsilon_2): \quad & \begin{cases} x_3 = \varepsilon_1 i x_1 = \varepsilon_2 i x_4, \\ x_0 = \varepsilon_1 i x_2. \end{cases} \end{aligned}$$

In particular all of the lines split over $\mathbb{Q}(i)$ and X contains precisely 8 lines that are defined over \mathbb{Q} . Let us write $\mathcal{G} = \text{Gal}(\mathbb{Q}(i)/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$ for the Galois group of the splitting field. As described in the book by Manin [20], if Λ denotes the class of a hyperplane section, then the geometric Picard group $\text{Pic}_{\overline{\mathbb{Q}}} X$ is given by $\{\Lambda, E_1, \dots, E_5\}$, where E_1, \dots, E_5 are the classes of a set of 5 lines in X that are mutually skew. Identifying divisors with their classes in $\text{Pic}_{\overline{\mathbb{Q}}} X$, it is easily checked that the lines

$$\begin{aligned} E_1 &= M_1(-1, 1), & E_2 &= M_1(1, -1), & E_3 &= M_2(-1, -1), \\ E_4 &= M_4(-1, -1), & E_5 &= M_4(1, 1), \end{aligned}$$

satisfy the desired property. Since E_1, E_2, E_3 are defined over \mathbb{Q} and E_4, E_5 are defined over $\mathbb{Q}(i)$, but are conjugate under the action of \mathcal{G} , so it follows that

$$\text{Pic } X = (\text{Pic}_{\overline{\mathbb{Q}}} X)^{\mathcal{G}} = \{\Lambda, E_1, E_2, E_3, E_4 + E_5\}.$$

This retrieves the information $\text{Pic } X \cong \mathbb{Z}^5$ that was recorded in the introduction. It now follows from taking $k = \mathbb{Q}$ in Table 1 in the appendix that

$$\alpha(X) = \frac{1}{36}. \tag{10.1}$$

The constant $\beta(X)$ is equal to the cardinality of $H^1(\mathbb{Q}, \text{Pic}_{\overline{\mathbb{Q}}} X)$, where $\text{Pic}_{\overline{\mathbb{Q}}} X$ is the geometric Picard group. In the present setting we have

$$\beta(X) = 1, \tag{10.2}$$

since X is \mathbb{Q} -birationally trivial.

We now turn to the value of the constant $\tau_H(X)$. For any place v of \mathbb{Q} let $\omega_{H,v}$ be the usual v -adic density of points on the locally compact space $X(\mathbb{Q}_v)$. The Tamagawa number associated to X and H is then given by

$$\tau_H(X) = \lim_{s \rightarrow 1} ((s-1)^5 L(s, \text{Pic}_{\overline{\mathbb{Q}}} X)) \omega_{H,\infty} \prod_p \frac{\omega_{H,p}}{L_p(1, \text{Pic}_{\overline{\mathbb{Q}}} X)}.$$

In the present setting it follows from our calculation of $\text{Pic}_{\overline{\mathbb{Q}}} X$ that

$$L(s, \text{Pic}_{\overline{\mathbb{Q}}} X) = \zeta_{\mathbb{Q}}(s)^4 \zeta_{\mathbb{Q}(i)}(s) = \zeta_{\mathbb{Q}}(s)^5 L(s, \chi),$$

where χ is the real non-principal character modulo 4, whence

$$\tau_H(X) = \frac{\pi}{4} \omega_{H,\infty} \prod_p \left(1 - \frac{1}{p}\right)^5 \left(1 - \frac{\chi(p)}{p}\right) \omega_{H,p} = \omega_{H,\infty} \prod_p \left(1 - \frac{1}{p}\right)^5 \omega_{H,p}.$$

Recall the definitions (2.6), (2.3) of g and $\overline{\varrho}_p^\dagger(\nu_1, \nu_2, \nu_3)$, respectively. We will show that

$$\omega_{H,p} = \kappa_p \left(1 - \frac{1}{p}\right)^{-1} \left(1 + \frac{1}{p}\right) \sum_{\nu \in \mathbb{Z}_{\geq 0}^3} g(p^{\nu_1 + \nu_2 + \nu_3}) \overline{\varrho}_p^\dagger(\nu_1, \nu_2, \nu_3), \quad (10.3)$$

with

$$\kappa_p := \begin{cases} 1, & \text{if } p > 2, \\ 4/3, & \text{if } p = 2. \end{cases} \quad (10.4)$$

Furthermore, we will demonstrate the equality

$$\omega_{H,\infty} = 16\sigma_\infty, \quad (10.5)$$

where σ_∞ is given by (9.4).

Subject to the proofs of (10.3) and (10.5), let us now verify that the constant in (9.5) coincides with that predicted by Peyre. Bringing together our expression for $\tau_H(X)$ with (10.1) and (10.2), we find that

$$c_{X,H} = \alpha(X)\beta(X)\tau_H(X) = \frac{16C^*\sigma_\infty}{27\zeta(2)},$$

in the notation of (2.4). This therefore confirms the constant in (9.5) is the same one that is predicted by Peyre.

10.1. Calculation of $\omega_{H,\infty}$. — In order to prove (10.5) we will adhere to the method outlined by Peyre [22]. Taking into account the fact that \mathbf{x} and $-\mathbf{x}$ represent the same point in $\mathbb{P}_{\overline{\mathbb{Q}}}^4$, the archimedean density of points on X is equal to

$$\begin{aligned} \omega_{H,\infty} &= \frac{1}{2} \int_{\{\mathbf{x} \in \mathbb{R}^5: \Phi_1(\mathbf{x}) = \Phi_2(\mathbf{x}) = 0, \|\mathbf{x}\| \leq 1\}} \omega_L(\mathbf{x}) \\ &= 8 \int_{\{\mathbf{x} \in \mathbb{R}_{\geq 0}^5: \Phi_1(\mathbf{x}) = \Phi_2(\mathbf{x}) = 0, \max\{x_0, x_1, x_2, x_3\} \leq 1\}} \omega_L(\mathbf{x}), \end{aligned}$$

where $\omega_L(\mathbf{x})$ is the Leray form, and we have used the same sort of symmetry arguments apparent in Lemma 6. It will be convenient to parametrise the points via the choice of variables x_0, x_1, x_2 . Observe that

$$\det \begin{pmatrix} \frac{\partial \Phi_1}{\partial x_3} & \frac{\partial \Phi_2}{\partial x_3} \\ \frac{\partial \Phi_1}{\partial x_4} & \frac{\partial \Phi_2}{\partial x_4} \end{pmatrix} = 4x_2x_4.$$

We will take the Leray form $\omega_L(\mathbf{x}) = (4x_2x_4)^{-1} dx_0 dx_1 dx_2$. Recall the definition (5.5) of p_u, q_u, r_u . We will pass from x_0, x_1, x_2 to (s, t, u) , with $t, u > 0$, via the change of variables

$$(x_0, x_1, x_2) = (p_u(s, \sqrt{|1-u^2|t}), uq_u(s, \sqrt{|1-u^2|t}), q_u(s, \sqrt{|1-u^2|t})).$$

The Jacobian of this transformation is calculated to be

$$8q_u(s, \sqrt{|1-u^2|t})r_u(s, \sqrt{|1-u^2|t}) = 8x_2x_4,$$

on noting that

$$(1-u^2)p_u(s, \sqrt{|1-u^2|t})^2 + (1+u^2)q_u(s, \sqrt{|1-u^2|t})^2 = 2r_u(s, \sqrt{|1-u^2|t})^2.$$

A modest pause for thought now reveals that (10.5) holds, as claimed.

10.2. Calculation of $\omega_{H,p}$. — It will ease notation if we write $\mathbb{Z}/p^k\mathbb{Z}$ throughout this section, for any prime power p^k . It remains to calculate the value of $\omega_{H,p} = \lim_{n \rightarrow \infty} p^{-3n} N(p^n)$, for any prime p , where $N(p^n)$ denotes the number of $\mathbf{x} \in (\mathbb{Z}/p^n)^5$ for which $\Phi_1(\mathbf{x}) \equiv \Phi_2(\mathbf{x}) \equiv 0 \pmod{p^n}$. The usual way of proceeding at this point would be to try and interpret $p^{-3n} N(p^n)$ as an explicit function of p^{-1} . This would then entail a parallel calculation of C^* in (2.4), in order to check that the values match. Instead we will try and calculate $N(p^n)$ by mimicking the steps taken in the proof of the theorem. Recall the definitions (1.2), (1.4) of Φ_1 and Φ_2 .

It is easy to check that $\omega_{H,p} = (1-1/p)^{-1} \omega_{H,p}^*$, where $\omega_{H,p}^* = \lim_{n \rightarrow \infty} p^{-3n} N^*(p^n)$, for any prime p , with

$$N^*(p^n) := \#\{\mathbf{x} \in (\mathbb{Z}/p^n)^5 : p \nmid \mathbf{x}, \Phi_1(\mathbf{x}) \equiv \Phi_2(\mathbf{x}) \equiv 0 \pmod{p^n}\}.$$

We will show that

$$\omega_{H,p}^* = \kappa_p \left(1 + \frac{1}{p}\right) \sum_{\nu \in \mathbb{Z}_{\geq 0}^3} g(p^{\nu_1+\nu_2+\nu_3}) \bar{g}_p^\dagger(\nu_1, \nu_2, \nu_3), \quad (10.6)$$

where κ_p is given by (10.4). This will clearly be enough to establish (10.3).

Let \mathbf{x} be any vector counted by $N^*(p^n)$. We claim that there are precisely $\varphi(p^n)$ choices of $(a, b, x, y, z) \in (\mathbb{Z}/p^n)^5$ such that $p \nmid (a, b)$, $p \nmid (x, y)$ and

$$\mathbf{x} \equiv (ax, by, ay, bx, z) \pmod{p^n}.$$

To see this we may suppose without loss of generality that $p \nmid x_0$. For any of $\varphi(p^n)$ choices of $a \in (\mathbb{Z}/p^n)^*$, the value of x is determined uniquely modulo p^n via the congruence $x_0 \equiv ax \pmod{p^n}$. But then values of b and y are also determined exactly modulo p^n , which therefore establishes the claim.

Recall the definition of $C_{a,b}$ from (1.6). By a convenient abuse of notation we will write $(x, y, z) \in C_{a,b}(\mathbb{Z}/p^n)$ to denote that the underlying quadratic polynomial is

congruent to zero modulo p^n . It now follows that

$$\begin{aligned} N^*(p^n) &= \frac{1}{\varphi(p^n)} \sum_{\substack{a,b \bmod p^n \\ p \nmid (a,b)}} \#\{(x,y,z) \in C_{a,b}(\mathbb{Z}/p^n) : p \nmid (x,y)\} \\ &= \frac{1}{\varphi(p^n)} \sum_{\nu_1, \nu_2, \nu_3 \geq 0} \sum_{\substack{a,b \bmod p^n \\ p \nmid (a,b) \\ p^{\nu_i} \parallel a + (-1)^i b \\ p^{\nu_3} \parallel a^2 + b^2}} \#\{(x,y,z) \in C_{a,b}(\mathbb{Z}/p^n) : p \nmid (x,y)\}, \end{aligned}$$

where the index i belongs to $\{1, 2\}$. Moreover, in view of the fact that (1.7) is always a solution of the relevant congruence modulo p^n we may assume in all of our calculations that the summand is always positive. Here we recall the convention that $p^\nu \parallel n$ if and only if $\nu_p(n) = \nu$.

For given $c, d \in \mathbb{Z}$, with $p \nmid cd$, and given $\mu, \nu \geq 0$, let us define

$$D_{\mu,\nu}^*(p^n) := p^{-2n} \#\left\{(x,y,z) \in (\mathbb{Z}/p^n)^3 : \begin{array}{l} p \nmid (x,y), \\ cp^\mu x^2 + dp^\nu y^2 \equiv 2z^2 \pmod{p^n} \end{array} \right\}. \quad (10.7)$$

It will be convenient to define $D_{\mu,\nu}(p^n)$ as for $D_{\mu,\nu}^*(p^n)$, but without the condition that $p \nmid (x,y)$. A little thought reveals that we have

$$D_{\mu,\nu}^*(p^n) = D_{\mu,\nu}(p^n) - p^{-1} D_{\mu,\nu}(p^{n-2}) \quad (10.8)$$

for each $n \geq 2$ and $\mu, \nu \geq 0$. Our calculation of $D_{\mu,\nu}^*(p^n)$ will depend intimately on the values of μ, ν and whether or not p is odd. We have collected together the necessary information in the following result.

Lemma 25. — *Let $p > 2$. Then we have*

$$D_{0,0}^*(p^n) = \left(1 - \frac{1}{p^2}\right), \quad (10.9)$$

and furthermore,

$$D_{\mu,0}^*(p^n) = \left(1 - \frac{1}{p}\right) \left(\mu \left(1 - \frac{1}{p}\right) + 1 + \frac{1}{p}\right) + O(p^{-(n-\mu-2)}). \quad (10.10)$$

if $\mu \geq 1$ and $(\frac{2d}{p}) = 1$. Let $p = 2$. Then we have

$$D_{\mu,\nu}^*(2^n) = \begin{cases} 1, & \text{if } \mu = \nu = 0 \text{ and } c + d \equiv 0, 2 \pmod{8}, \\ \mu, & \text{if } \mu \geq 3, \nu = 1 \text{ and } 2^{\mu-1}c + d \equiv 1 \pmod{8}. \end{cases} \quad (10.11)$$

We postpone the proof of this result until later. For the moment we return to the expression for $N^*(p^n)$ that we are trying to evaluate. For any prime p and $\nu_1, \nu_2, \nu_3 \geq 0$ recall the definitions (2.2) and (2.3) of $\varrho_p^\dagger(\nu_1, \nu_2, \nu_3)$ and $\bar{\varrho}_p^\dagger(\nu_1, \nu_2, \nu_3)$, respectively. In particular it is not hard to see that $\varrho_p^\dagger(\nu_1, \nu_2, \nu_3) = O(p^{\nu_1 + \nu_2 + \nu_3 + 2})$.

We have

$$N^*(p^n) = \frac{p^{4n}}{\varphi(p^n)} \sum_{\nu_1, \nu_2, \nu_3 \geq 0} \frac{1}{p^{2\nu}} \sum_{\substack{a, b \bmod p^\nu \\ p \nmid (a, b) \\ p^{\nu_i} \parallel a + (-1)^i b \\ p^{\nu_3} \parallel a^2 + b^2}} \frac{\#\{(x, y, z) \in C_{a, b}(\mathbb{Z}/p^n) : p \nmid (x, y)\}}{p^{2n}},$$

where we have written $\nu = \nu_1 + \nu_2 + \nu_3 + 1$ for convenience.

Beginning with the case $p > 2$ let us write

$$a^2 - b^2 = p^{\nu_1 + \nu_2} c, \quad a^2 + b^2 = p^{\nu_3} d,$$

for c, d coprime to p . Since $p \nmid (a, b)$ it clearly follows that at most one of ν_1, ν_2, ν_3 can be non-zero. When they are all zero (10.9) reveals that the summand here is $(1 - 1/p^2)$. When $\nu_1 \geq 1$ and $\nu_2 = \nu_3 = 0$ we know that $2d$ is a quadratic residue modulo p since $C_{a, b}(\mathbb{Z}/p^n)$ is always non-empty. But then we may apply (10.10) to estimate the summand. This same estimate also covers the case in which ν_2 or ν_3 is positive. When $p > 2$ we therefore deduce that

$$\lim_{n \rightarrow \infty} \frac{N^*(p^n)}{p^{3n}} = \sum_{\nu_1, \nu_2, \nu_3 \geq 0} \left((\nu - 1) \left(1 - \frac{1}{p}\right) + 1 + \frac{1}{p} \right) \bar{\varrho}_p^\dagger(\nu_1, \nu_2, \nu_3)$$

with $\nu = \nu_1 + \nu_2 + \nu_3 + 1$. Recall the definition (2.6) of g for odd prime powers. It is not hard to see that

$$\left(1 + \frac{1}{p}\right) g(p^{\nu-1}) = (\nu - 1) \left(1 - \frac{1}{p}\right) + 1 + \frac{1}{p}.$$

It therefore follows that (10.6) holds when $p > 2$, with $\kappa_p = 1$.

It remains to deal with the case $p = 2$. Arguing as for $p > 2$ we write

$$a^2 - b^2 = 2^{\nu_1 + \nu_2} c, \quad a^2 + b^2 = 2^{\nu_3} d,$$

for odd c, d . Since $2 \nmid (a, b)$ it now follows that either $\nu_1 = \nu_2 = \nu_3 = 0$, or else $\nu_1 + \nu_2 \geq 3$ and $\nu_3 = 1$. In the latter case we have $\nu_1 \geq 2$ and $\nu_2 = 1$, or $\nu_2 \geq 2$ and $\nu_1 = 1$. In every case we know that $C_{a, b}(\mathbb{Z}/2^n)$ is non-empty, and in fact has a solution $(x, y, z) \equiv (1, 1, a) \bmod 2^n$. Suppose first that $\nu_1 = \nu_2 = \nu_3 = 0$. Then precisely one of a, b must be even, and we automatically have $c + d \equiv 0, 2 \bmod 8$. But then (10.11) implies that the innermost summand in our expression for $N^*(2^n)$ is 1. When $\nu_1 \geq 2$ and $\nu_2 = \nu_3 = 1$ we may apply (10.11) with $\mu = \nu_1 + 1$ to deduce that the summand is $\nu_1 + 1 = \nu_1 + \nu_2 + \nu_3 - 1$. The same applies when $\nu_2 \geq 2$ and $\nu_1 = \nu_3 = 1$. Putting this all together we deduce that

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{N^*(2^n)}{2^{3n}} &= 2 \left(\bar{\varrho}_2^\dagger(0, 0, 0) + \sum_{\substack{\nu_1, \nu_2, \nu_3 \geq 0 \\ \nu_1 + \nu_2 + \nu_3 \geq 1}} (\nu_1 + \nu_2 + \nu_3 - 1) \bar{\varrho}_2^\dagger(\nu_1, \nu_2, \nu_3) \right) \\ &= 2 \sum_{\nu_1, \nu_2, \nu_3 \geq 0} \max\{1, \nu_1 + \nu_2 + \nu_3 - 1\} \bar{\varrho}_2^\dagger(\nu_1, \nu_2, \nu_3). \end{aligned}$$

Recall the definition (2.6) of g for even prime powers. It is now clear that (10.6) holds when $p = 2$, with $\kappa_2 = 4/3$.

Proof of Lemma 25. — Throughout our proof of the lemma we can and will assume that n is large compared with μ and ν .

Assume that $p > 2$ and let $\mu = \nu = 0$. It follows from Hensel's lemma that

$$D_{0,0}^*(p^n) = D_{0,0}^*(p) = p^{-2}M_{0,0}^*(p),$$

where $M_{0,0}^*(p)$ is the number of $(x, y, z) \in (\mathbb{Z}/p)^3$ such that $p \nmid (x, y)$ and $cx^2 + dy^2 \equiv 2z^2 \pmod{p}$. The latter can easily be calculated using Gauss sums with the outcome that $M_{0,0}^*(p) = p^2 - 1$. Hence (10.9) is now obvious.

We now turn to the calculation of $D_{\mu,0}^*(p^n)$, for $\mu \geq 1$, for which we will assume that $(\frac{2d}{p}) = 1$, so that $2d$ can be lifted to a quadratic residue modulo p^n . Hence

$$D_{\mu,0}^*(p^n) = p^{-2n} \# \{ (x, y, z) \in (\mathbb{Z}/p^n)^3 : p \nmid (x, y), 2cp^\mu x^2 + y^2 \equiv z^2 \pmod{p^n} \}.$$

Similarly for $D_{\mu,0}(p^n)$. The contribution to the right hand side from those x, y, z for which $p \nmid y$ is $2(1 - 1/p)$. This follows from an argument based on Hensel's lemma and an easy analysis of the corresponding congruence modulo p . The contribution from those x, y, z for which $p \mid y$ is 0 if $\mu = 1$ and

$$p^{-2n} \# \{ (x, y, z) \in (\mathbb{Z}/p^n)^* \times (\mathbb{Z}/p^{n-1})^2 : 2cp^\mu x^2 + p^2 y^2 \equiv p^2 z^2 \pmod{p^n} \},$$

if $\mu \geq 2$. Assuming $\mu \geq 2$ we therefore obtain the contribution

$$\begin{aligned} & p^{4-2n} \# \{ (x, y, z) \in (\mathbb{Z}/p^{n-2})^3 : 2cp^{\mu-2} x^2 + y^2 \equiv z^2 \pmod{p^{n-2}} \} \\ & - p^{3-2n} \# \{ (x, y, z) \in (\mathbb{Z}/p^{n-2})^3 : 2cp^\mu x^2 + y^2 \equiv z^2 \pmod{p^{n-2}} \}. \end{aligned}$$

But this is just $D_{\mu-2,0}(p^{n-2}) - p^{-1}D_{\mu,0}(p^{n-2})$. It therefore follows that

$$D_{\mu,0}^*(p^n) = \begin{cases} 2(1 - 1/p), & \text{if } \mu = 1, \\ 2(1 - 1/p) + D_{\mu-2,0}(p^{n-2}) - p^{-1}D_{\mu,0}(p^{n-2}), & \text{if } \mu \geq 2. \end{cases}$$

Recalling (10.8), we have

$$D_{1,0}(p^n) = D_{1,0}^*(p^n) + p^{-1}D_{1,0}(p^{n-2}) = 2(1 - 1/p) + p^{-1}D_{1,0}(p^{n-2}),$$

and

$$D_{\mu,0}(p^n) = 2(1 - 1/p) + D_{\mu-2,0}(p^{n-2}),$$

if $\mu \geq 2$. In view of the fact that $D_{1,0}(p) = 2$ and $D_{1,0}(1) = 1$, we easily deduce that

$$D_{1,0}(p^n) = 2 + O(p^{-(n-2)}).$$

Moreover, on noting that $D_{0,0}(p) = 2 - 1/p^2$ and $D_{0,0}(1) = 1$, (10.8) and (10.9) give

$$D_{0,0}(p^n) = 1 - 1/p^2 + p^{-1}D_{0,0}(p^{n-2}) = 1 + 1/p + O(p^{-(n-2)}).$$

Armed with these expression we can conclude that (10.10) holds, as claimed.

We now examine the case $p = 2$. Recall the definition (10.7) of $D_{\mu,\nu}^*(2^n)$ and the corresponding definition of $D_{\mu,\nu}(2^n)$. Let $\mu = \nu = 0$, under which hypothesis we will assume that

$$c + d \equiv 0, 2 \pmod{8}. \quad (10.12)$$

Our argument will differ according to which of 0 or 2 it is that $c + d$ is congruent to modulo 8. Beginning with the case $c + d \equiv 2 \pmod{8}$, it is not hard to see that x, y, z

are all necessarily odd in the definition of $D_{0,0}^*(2^n)$. Writing $x = 1 + 2x', y = 1 + 2y'$ and $z = 1 + 2z'$ we deduce that

$$D_{0,0}^*(2^n) = 2^{-2n} \# \{ (x', y', z') \in (\mathbb{Z}/2^{n-1})^3 : f(x', y') \equiv z' + z'^2 \pmod{2^{n-2}} \},$$

where we have set $f(x', y') = 2e + cx'(1 + x') + dy'(1 + y')$ and $2e = (c + d - 2)/4$. For any $a \in \mathbb{Z}$ and any $m \geq 1$, let us denote by $S_a(2^m)$ the number of $x \in \mathbb{Z}/2^m$ for which $x(x + 1) \equiv a \pmod{2^m}$. An easy lifting argument reveals that

$$S_a(2^m) = S_a(2) = \begin{cases} 0, & \text{if } 2 \nmid a, \\ 2, & \text{if } 2 \mid a. \end{cases}$$

Varying x', y' and applying the latter equality to estimate the number of relevant z' , we may now conclude that $D_{0,0}^*(2^n) = 2^{-2n} \cdot 2^{n-1} \cdot 2^{n-1} \cdot 4 = 1$ when $c + d \equiv 2 \pmod{8}$. Turning to the case $c + d \equiv 0 \pmod{8}$, we conclude that x, y are both necessarily odd in the definition of $D_{0,0}^*(2^n)$ and z is even. Writing $x = 1 + 2x', y = 1 + 2y'$ and $z = 2z'$ we deduce that

$$D_{0,0}^*(2^n) = 2^{-2n} \# \{ (x', y', z') \in (\mathbb{Z}/2^{n-1})^3 : f(x', y') \equiv 2z'^2 \pmod{2^{n-2}} \},$$

where $f(x', y')$ is as above but this time with $2e = (c + d)/4$. We now conclude as previously. Altogether we have shown that (10.11) holds with $\mu = \nu = 0$, under the assumption that (10.12) holds.

We proceed to consider $D_{\mu,1}^*(2^n)$, for $\mu \geq 3$, observing that

$$\begin{aligned} D_{\mu,1}^*(2^n) &= 2^{-2n} \# \{ (x, y, z) \in (\mathbb{Z}/2^n)^3 : 2 \nmid (x, y), \ 2^\mu cx^2 + 2dy^2 \equiv 2z^2 \pmod{2^n} \} \\ &= 2^{3-2n} \# \left\{ (x, y, z) \in (\mathbb{Z}/2^{n-1})^3 : \begin{array}{l} 2 \nmid (x, y), \\ 2^{\mu-1}cx^2 + dy^2 \equiv z^2 \pmod{2^{n-1}} \end{array} \right\}. \end{aligned}$$

We will make the assumption that

$$2^{\mu-1}c + d \equiv 1 \pmod{8}. \quad (10.13)$$

For any odd $a \in \mathbb{Z}$ and any $m \geq 3$, let $T_a(2^m)$ denote the number of $x \in \mathbb{Z}/2^m$ for which $x^2 \equiv a \pmod{2^m}$. It is an easy exercise to check that

$$T_a(2^m) = \begin{cases} 0, & \text{if } 2 \nmid a \text{ and } a \not\equiv 1 \pmod{8}, \\ 4, & \text{if } a \equiv 1 \pmod{8}, \end{cases}$$

for $m \geq 3$. This is implied by our previous bound for $S_a(2^m)$, for example.

Let us consider the contribution from those x, y, z for which $2 \nmid y$, appealing to (10.13) where necessary. For each $x, y \in \mathbb{Z}/2^{n-1}$ such that $2 \nmid y$ and $2^{\mu-1}cx^2 + dy^2 \equiv 1 \pmod{8}$, our calculation of $T_a(2^{n-1})$ shows that there are exactly 4 choices for z . Altogether, we therefore obtain the contribution $2^{3-2n} \cdot 2^{n-2} \cdot 2^{n-2} \cdot 4 = 2$ when $\mu = 3$, and $2^{3-2n} \cdot 2^{n-1} \cdot 2^{n-2} \cdot 4 = 4$ when $\mu \geq 4$. The contribution from those x, y, z for which $2 \mid y$ is clearly

$$2^{7-2n} \# \{ (x, y, z) \in (\mathbb{Z}/2^{n-3})^3 : 2 \nmid x, \ 2^{\mu-3}cx^2 + dy^2 \equiv z^2 \pmod{2^{n-3}} \} = N_\mu,$$

say. We will show that

$$N_\mu = \begin{cases} 1, & \text{if } \mu = 3, \\ \mu - 4, & \text{if } \mu \geq 4. \end{cases} \quad (10.14)$$

Together with the contribution from the previous treatment, this will suffice to show that (10.11) holds when $\mu \geq 3$ and $\nu = 1$, under the hypothesis (10.13).

Turning to the proof of (10.14), suppose first that $\mu = 3$. Then $d \equiv 5 \pmod{8}$, by (10.13), and

$$N_3 = 2^{7-2n} \# \{ (x, y, z) \in (\mathbb{Z}/2^{n-3})^3 : 2 \nmid x, cx^2 + dy^2 \equiv z^2 \pmod{2^{n-3}} \}.$$

When $2 \mid y$ we note that for any $x, y \in \mathbb{Z}/2^{n-3}$ such that $2 \nmid x$ and $cx^2 + dy^2 \equiv 1 \pmod{8}$, there are 4 choices for z . The condition modulo 8 is clearly equivalent to $c + 5y^2 \equiv 1 \pmod{8}$. In particular, assuming that $c \equiv 1 \pmod{4}$, we get an overall contribution of $2^{7-2n} \cdot 2^{n-4} \cdot 2^{n-5} \cdot 4 = 1$. Alternatively, the case $2 \nmid y$ contributes the same amount via a similar argument, but this time under the hypothesis $c \equiv 3 \pmod{4}$. Together this confirms that N_3 is indeed 1.

The case $\mu = 4$ is easily seen to be impossible, whence $N_4 = 0$. Turning to the case $\mu = 5$, for which (10.13) implies that $d \equiv 1 \pmod{8}$, we deduce that both y and z must be even. Hence it follows that

$$N_5 = 2^{11-2n} \# \{ (x, y, z) \in (\mathbb{Z}/2^{n-5})^3 : 2 \nmid x, cx^2 + y^2 \equiv z^2 \pmod{2^{n-5}} \}.$$

The argument now goes as for the case $\mu = 3$, with a separate analysis of the cases $2 \mid y$ and $2 \nmid y$. Each contributes 1 to N_5 , but only one such case arises according to whether c is congruent to 1 or 3 modulo 4. Hence $N_5 = 1$.

Finally let us turn to the case $\mu \geq 6$, for which we still have $d \equiv 1 \pmod{8}$ by (10.13). It will now be convenient to write $N_\mu = N_\mu(n)$ to indicate the dependence on n in the definition. Now either y, z are both odd or they are both even. Since $2^{\mu-3}cx^2 + dy^2 \equiv 1 \pmod{8}$ whenever $\mu \geq 6$ and y is odd, so it follows that the contribution to $N_\mu(n)$ from odd y is $2^{7-2n} \cdot 2^{2(n-4)} \cdot 4 = 2$, by our expression for $T_a(2^m)$. The contribution from even y is easily seen to be $N_{\mu-2}(n-2)$. But then induction on μ reveals that the contribution from even y is $\mu - 6$. Together the two contributions suffice to ensure that (10.14) holds when $\mu \geq 6$. \square

Appendix by U. Derenthal

THE NEF CONE VOLUME IN DEGREES 3 AND 4

The nef cone volume. — For $d \in \{3, \dots, 6\}$, let X be a non-singular del Pezzo surface of degree d , defined over a perfect field k with an algebraic closure \bar{k} . We want to study the nef cone volume of X , defined by Peyre [22, Definition 2.4] to be

$$\alpha(X) = \text{vol}\{x \in \text{Nef}(X) \mid (x, -K_X) = 1\},$$

where $-K_X$ is the anticanonical class, $\text{Nef}(X)$ is the dual cone with respect to the intersection form (\cdot, \cdot) of the cone of classes of effective divisors in $\text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$, and the volume form is described in [23, Section 6], [12, Section 2]. Assume $X(k) \neq \emptyset$. Then $\text{Pic}(X) = \text{Pic}(X_{\bar{k}})^{G_k}$, where $X_{\bar{k}} = X \times_k \bar{k}$ and $G_k = \text{Gal}(\bar{k}/k)$ is the absolute Galois group.

By [12, Section 7], $\alpha(X)$ depends only on the action of G_k on the lines on X . More precisely, this action of G_k factorises through the Weyl group $W(R)$ associated to the root system

$$R = \{E \in \text{Pic}(X_{\bar{k}}) \mid (E, E) = -2, (E, -K_X) = 0\}.$$

By [12, Corollary 7.3], the constant $\alpha(X)$ depends only on the conjugacy class of the image H of G_k as a subgroup of $W(R)$.

The conjugacy classes of subgroups of $W(R)$ can be listed using **Magma** [2], for example. By the results of [12, Section 7], the computation of $\alpha(X)$ corresponding to a given subgroup H is completely straightforward. One determines the sum of elements of each orbit under the action of H on the set of classes of lines in $\text{Pic}(X_{\bar{k}})$. By [12, Corollary 6.4], these sums generate the cone of effective divisors in $\text{Pic}(X)$. Using [14], for example, one computes the dual of this cone and then the volume $\alpha(X)$.

Results. — Our goal is to compute $\alpha(X)$ corresponding to all conjugacy classes of subgroups H of $W(R)$, but to organise the results in a way that allows us to read off $\alpha(X)$ for a given del Pezzo surface X without determining the corresponding H .

For $d = 5, 6$, the values of $\alpha(X)$ corresponding to all conjugacy classes of subgroups of $W(\mathbf{A}_4)$ resp. $W(\mathbf{A}_2 + \mathbf{A}_1)$ are listed in [12, Tables 6, 8].

For $d = 4$, it turns out that $\alpha(X)$ depends only on the rank of $\text{Pic}(X)$ and, if $\text{rk Pic}(X) \in \{2, 3, 4\}$, the number of k -rational lines on X . Even though there are 197 conjugacy classes of subgroups of $W(\mathbf{D}_5)$, we get just nine cases that can be found in Table 1.

$\text{rk Pic}(X)$	k -rational lines	$\alpha(X)$
6	16	1/180
5	8	1/36
4	4	1/9
4	0	1/6
3	2	1/3
3	0	1/2
2	1	2/3
2	0	1
1	0	1

TABLE 1. $\alpha(X)$ for quartic surfaces X

For $d = 3$, the Weyl group $W(\mathbf{E}_6)$ has 350 conjugacy classes of subgroups. Here, the value of $\alpha(X)$ depends not only on $\text{rk Pic}(X)$ and the number of k -rational lines on $X_{\bar{k}}$. In most cases, it is enough to consider the following additional information on X : if there are n_i orbits of length a_i , for $i \in \{1, \dots, t\}$, then we denote this orbit structure by $(a_1^{n_1}, \dots, a_t^{n_t})$. Note that $n_1 a_1 + \dots + n_t a_t$ is the number of lines on $X_{\bar{k}}$. The values of $\alpha(X)$ for X of degree 3 can be found in Table 2. Only if this orbit structure is $(3^3, 6^3)$ (i.e., there are three orbits of length 3 and three orbits of length 6), this does not yet determine $\alpha(X)$ (marked by $*$ in the last column of Table 2). In this case,

rk Pic(X)	k -ratl. lines	orbit structure	$\alpha(X)$
7	27		1/120
6	15		1/30
5	7		1/8
5	9		5/48
4	3	$(1^3, 2^{12}), (1^3, 2^8, 4^2), (1^3, 2^6, 4^3)$	7/18
4	3	$(1^3, 2^3, 3^4, 6^1)$	3/8
4	5		5/18
3	0		1
3	1	$(1^1, 2^2, 3^2, 4^1, 6^2), (1^1, 2^4, 3^2, 6^2)$	1
3	1	$(1^1, 2^3, 4^5), (1^1, 2^3, 4^3, 8^1), (1^1, 2^5, 4^2, 8^1),$ $(1^1, 2^5, 4^4), (1^1, 2^2, 4^2, 6^1, 8^1)$	5/6
3	2		17/24
3	3		1/2
2	0	$(3^3, 6^3)$	*
2	0	$(3^2, 6^2, 9^1), (3^5, 6^2)$	2
2	0	$(2^1, 5^3, 10^1), (2^1, 5^1, 10^2)$	3/2
2	0	$(3^1, 6^4), (3^1, 6^2, 12^1), (6^3, 9^1), (6^2, 15^1)$	4/3
2	1		1
1	0		1

TABLE 2. $\alpha(X)$ for cubic surfaces X

we obtain $\alpha(X)$ as follows. Compute the sums of elements of each of these six orbits. If this gives six different elements of $\text{Pic}(X_{\bar{k}})$, then $\alpha(X) = 2$; otherwise, these sums give four different elements of $\text{Pic}(X_{\bar{k}})$, and $\alpha(X) = 4/3$.

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April 2, 2010

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